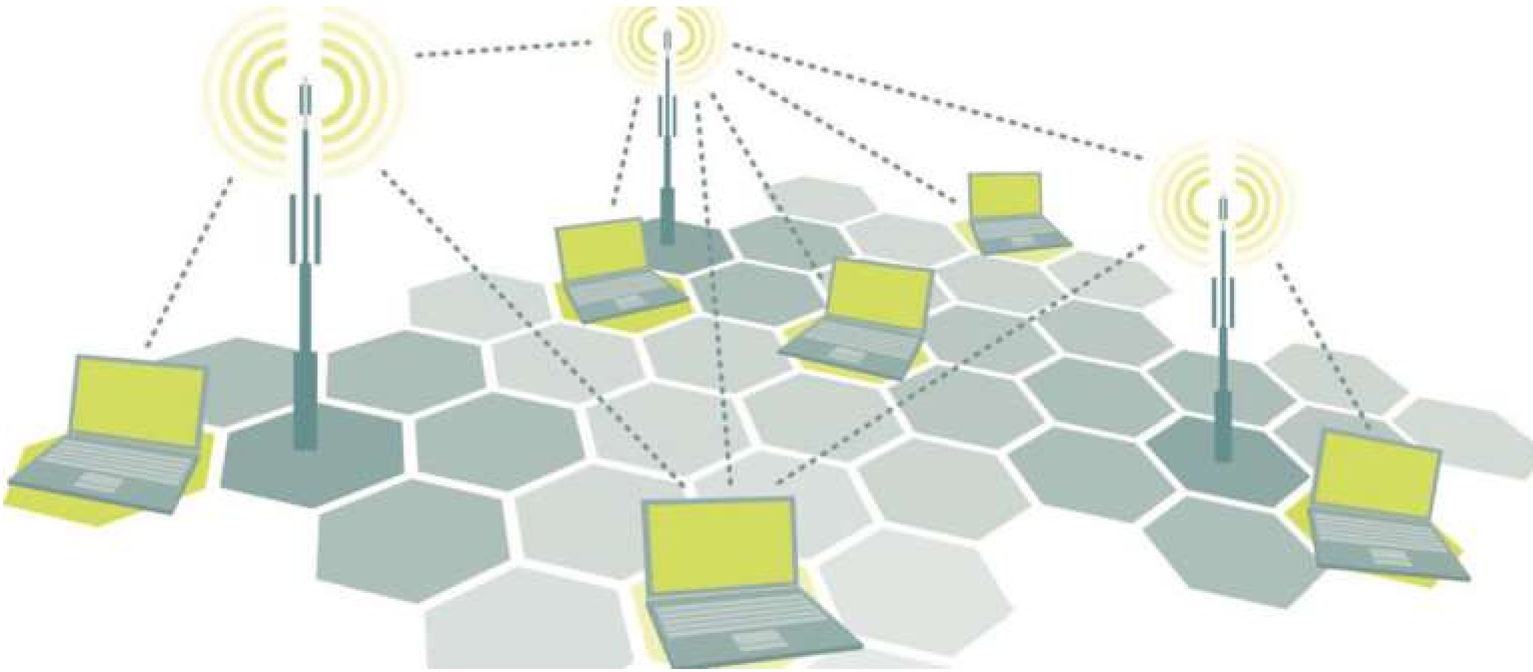
Lecture 5

Communication Theory



Fundamentals of Signals and Systems

Dr. Hany Ahmed

Introduction

Example 1.7 Discrete-Time Sinusoidal Signal

A pair of sinusoidal signals with a common angular frequency is defined by

$x_1[n] = \sin[5\pi n]$ **and** $x_2[n] = \sqrt{3}\cos[5\pi n]$

(a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find their common fundamental period.

(b) Express the composite sinusoidal signal

$y[n] = x_1[n] + x_2[n]$

In the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

<Sol.>

(a) Angular frequency of both $x_1[n]$ and $x_2[n]$:

$\Omega = 5\pi \quad \text{radians/cycle}$ ⇒ $N = \dfrac{2\pi m}{\Omega} = \dfrac{2\pi m}{5\pi} = \dfrac{2m}{5}$

Fundamenta frequency o $x[n]$: $\Omega = \dfrac{2\pi}{N}$

⇒ **This can be only for m = 5, 10, 15, …, which results in N = 2, 4, 6, …**

(b) Trigonometric identity:

$$A\cos(\Omega n + \phi) = A\cos(\Omega n)\cos(\phi) - A\sin(\Omega n)\sin(\phi)$$

Let $\Omega = 5\pi$, then compare $x_1[n] + x_2[n]$ with the above equation to obtain that

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$$A\sin(\phi) = -1 \quad \text{and} \quad A\cos(\phi) = \sqrt{3}$$

⇨ $$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{\text{amplitude of } x_1[n]}{\text{amplitude of } x_2[n]} = \frac{-1}{\sqrt{3}}$$ ⇨ $\phi = -\pi/6$

⇨ $$A\sin(\phi) = -1$$

⇨ $$A = \frac{-1}{\sin(-\pi/6)} = 2$$ **Accordingly, we may express *y*[*n*] as**

$$y[n] = 2\cos\left(5\pi n - \frac{\pi}{6}\right)$$

★ 1.6.3 Relation Between Sinusoidal and Complex Exponential Signals

1. Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$ **(1.41)**

Complex exponential signal: $B = Ae^{j\phi}$ **(1.42)**

$x(t) = A\cos(\omega t + \phi)$ **(1.35)**

⇨ $A\cos(\omega t + \phi) = \text{Re}\{Be^{j\omega t}\}$ **(1.42)**

$$\begin{aligned} &Be^{j\omega t} \\ &= Ae^{j\phi}e^{j\omega t} \\ &= Ae^{j(\phi+\omega t)} \\ &= A\cos(\omega t + \phi) + jA\sin(\omega t + \phi) \end{aligned}$$

Introduction

◇ **Continuous-time signal in terms of sine function:**

$$x(t) = A\sin(\omega t + \phi) \quad \textbf{(1.44)}$$

$$A\sin(\omega t + \phi) = \mathrm{Im}\{Be^{j\omega t}\} \quad \textbf{(1.45)}$$

2. Discrete-time case:

$$A\cos(\Omega n + \phi) = \mathrm{Re}\{Be^{j\Omega n}\} \quad \textbf{(1.46)}$$ **and** $$A\sin(\Omega n + \phi) = \mathrm{Im}\{Be^{j\Omega n}\} \quad \textbf{(1.47)}$$

3. Two-dimensional representation of the complex exponential $e^{j\Omega n}$ for $\Omega = \pi/4$ and n = 0, 1, 2, …, 7. : Fig. 1.34.

Projection on real axis: cos(Ω*n*);
Projection on imaginary axis: sin(Ω*n*)

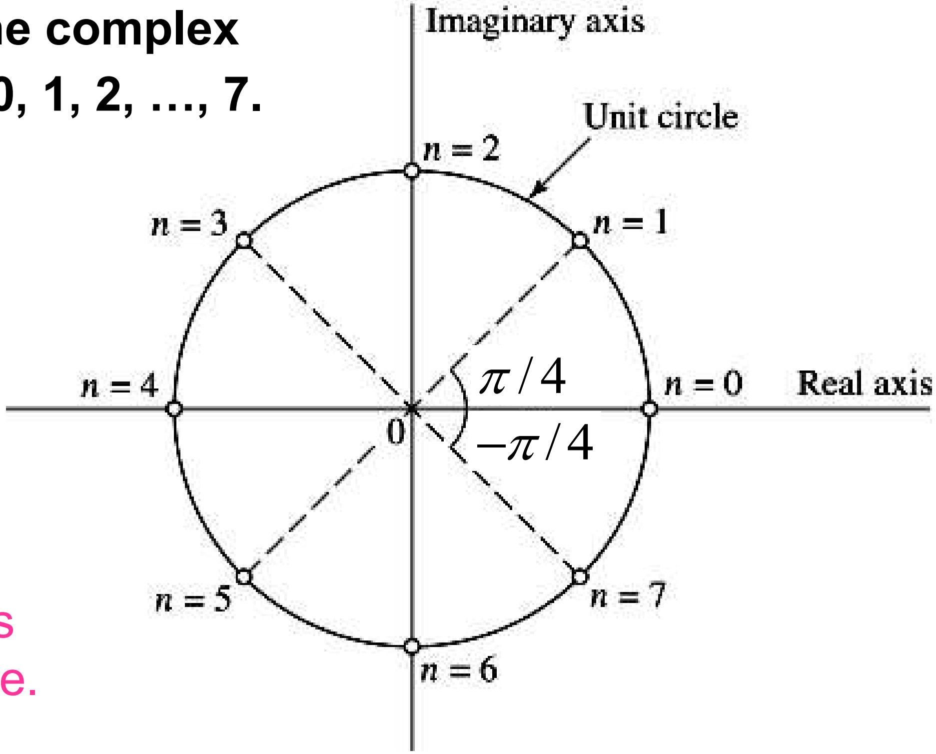


Figure 1.34 (p. 41)
Complex plane, showing eight points uniformly distributed on the unit circle.

Introduction

★ **1.6.4 Exponential Damped Sinusoidal Signals**

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \phi), \quad \alpha > 0 \qquad \textbf{(1.48)}$$

Example for A = 60, α = 6, and ϕ = 0: Fig.1.35.

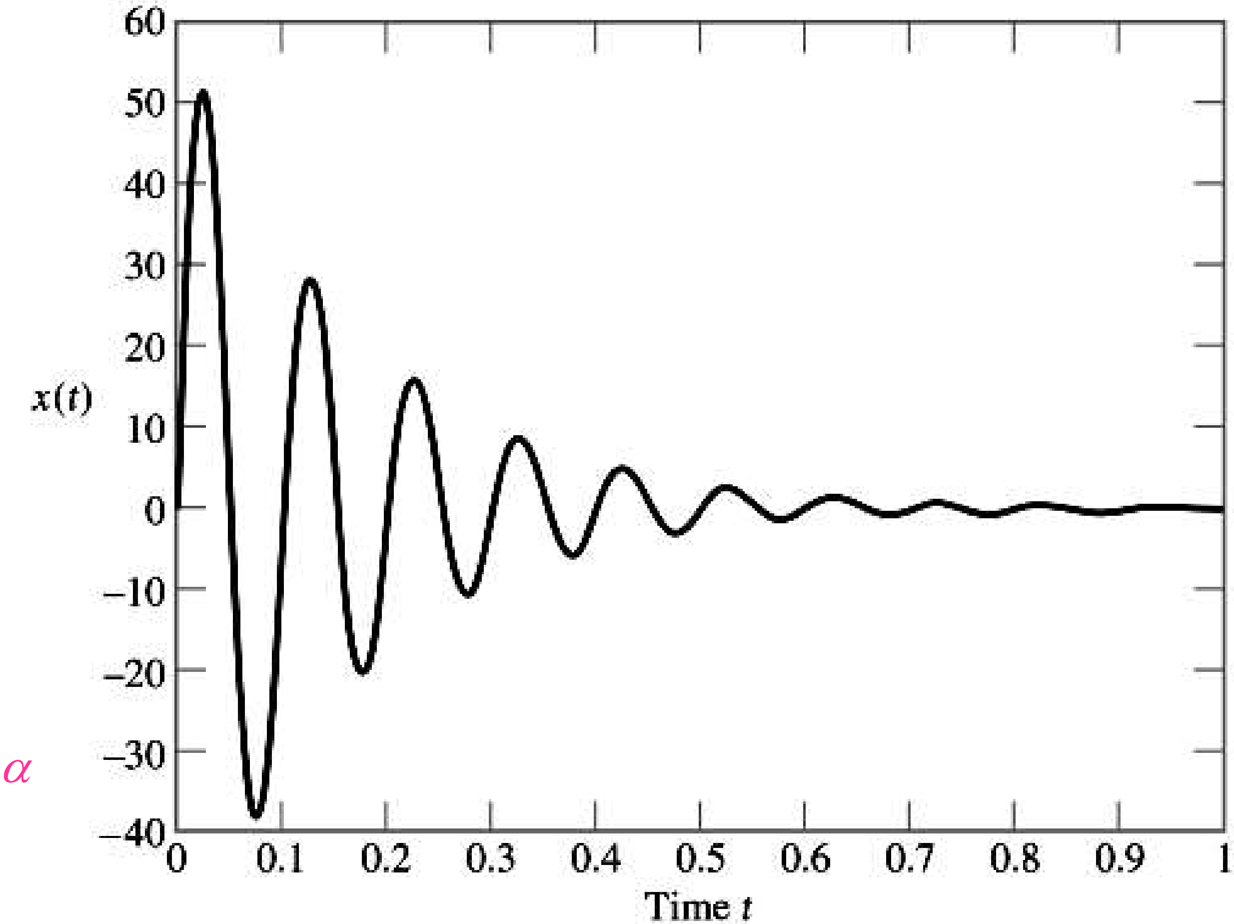


Figure 1.35 (p. 41) Exponentially damped sinusoidal signal $Ae^{-\alpha t}\sin(\omega t)$, with A = 60 and α = 6.

Introduction

Ex. Generation of an exponential damped sinusoidal signal
$\Rightarrow$ **Fig. 1-36.**

Circuit Eq.: $$C\frac{d}{dt}v(t)+\frac{1}{R}v(t)+\frac{1}{L}\int_{-\infty}^{t}v(\tau)d\tau=0 \quad (1.49)$$

$$v(t)=V_0e^{-t/(2CR)}\cos(\omega_0 t) \quad t\geq 0 \quad (1.50)$$

where $$\omega_0=\sqrt{\frac{1}{LC}-\frac{1}{4C^2R^2}} \quad (1.51)$$

$$R>\sqrt{L/(4C)}$$

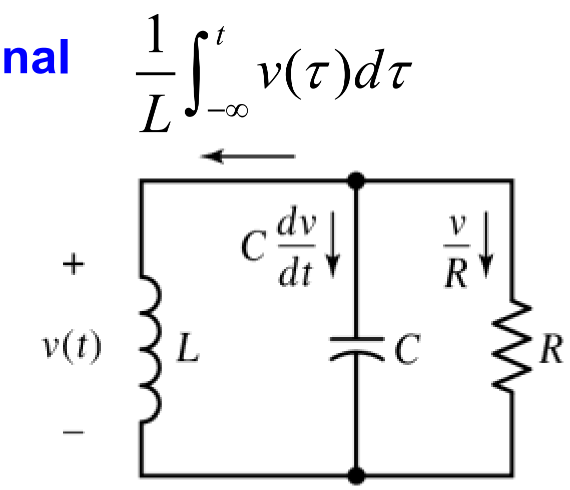


Figure 1.36 (p. 42)
Parallel *LRC*, circuit, with inductor *L*, capacitor *C*, and resistor *R* all assumed to be ideal.

Comparing Eq. (1.50) and (1.48), we have

$$A=V_0,\quad \alpha=1/(2CR),\quad \omega=\omega_0,\quad \text{and}\quad \phi=\pi/2$$

◆ **Discrete-time case:**

$$x[n]=Br^n\sin[\Omega n+\phi] \quad (1.52)$$

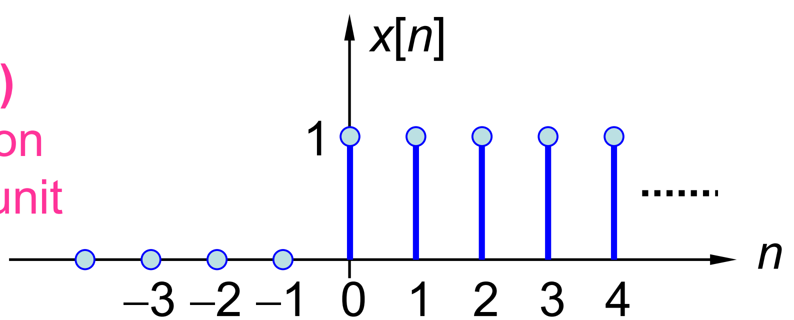
★ **1.6.5 Step Function**

◆ **Discrete-time case:**

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases} \quad (1.53)$$

Fig. 1-37.

Figure 1.37 (p. 43)
Discrete-time version of step function of unit amplitude.



Introduction

◆ **Continuous-time case:**

$$u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases} \quad (1.54)$$

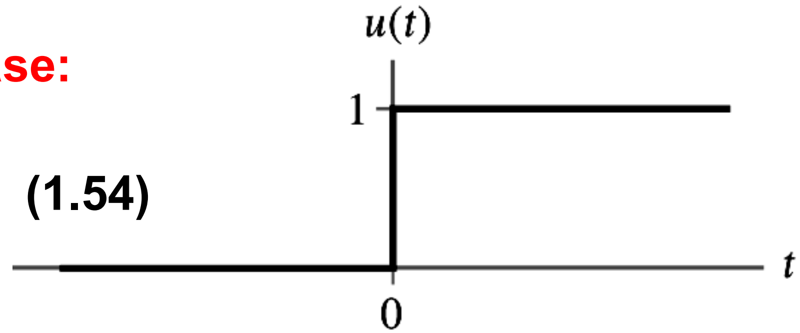


Figure 1.38 (p. 44) Continuous-time version of the unit-step function of unit amplitude.

Example 1.8 Rectangular Pulse

Consider the rectangular pulse *x*(*t*) shown in Fig. 1.39 (a). This pulse has an amplitude *A* and duration of 1 second. Express *x*(*t*) as a weighted sum of two step functions.

<Sol.>

1. Rectangular pulse *x*(*t*):

$$x(t)=\begin{cases}A, & 0\le|t|<0.5\\ 0, & |t|>0.5\end{cases} \quad (1.55)$$

$$\Rightarrow \quad x(t)=Au\left(t+\frac{1}{2}\right)-Au\left(t-\frac{1}{2}\right) \quad (1.56)$$

Example 1.9 *RC* Circuit

Find the response *v*(*t*) of *RC* circuit shown in Fig. 1.40 (a).

<Sol.>

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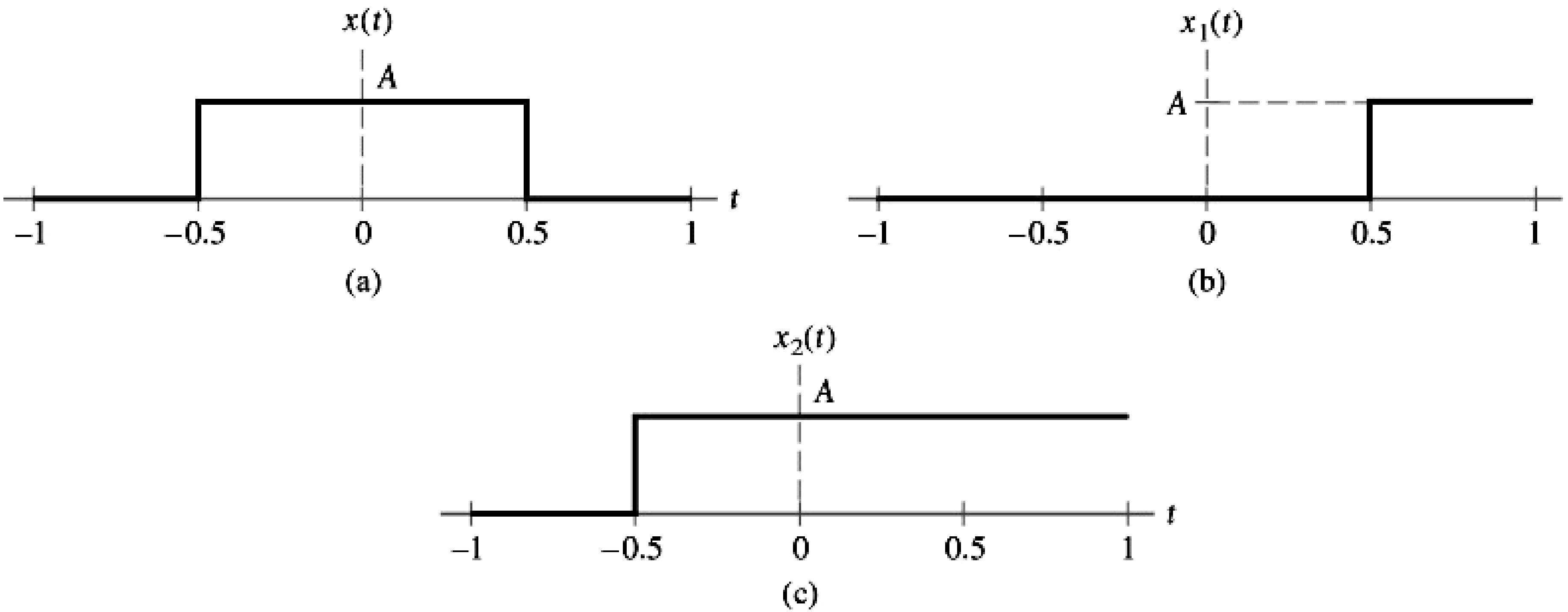

Figure 1.39 (p. 44)

(a) Rectangular pulse $x(t)$ of amplitude A and duration of 1 s, symmetric about the origin. (b) Representation of $x(t)$ as the difference of two step functions of amplitude A, with one step function shifted to the left by ½ and the other shifted to the right by ½; the two shifted signals are denoted by $x_1(t)$ and $x_2(t)$, respectively. Note that $x(t) = x_1(t) - x_2(t)$.

Introduction

★ **1.6.6 Impulse Function**

◆ **Discrete-time case:**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \qquad (1.58)$$

Fig. 1.41

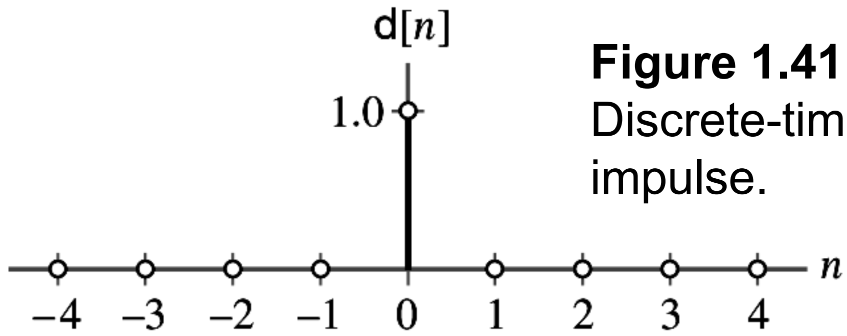

Figure 1.41 (p. 46)
Discrete-time form of impulse.

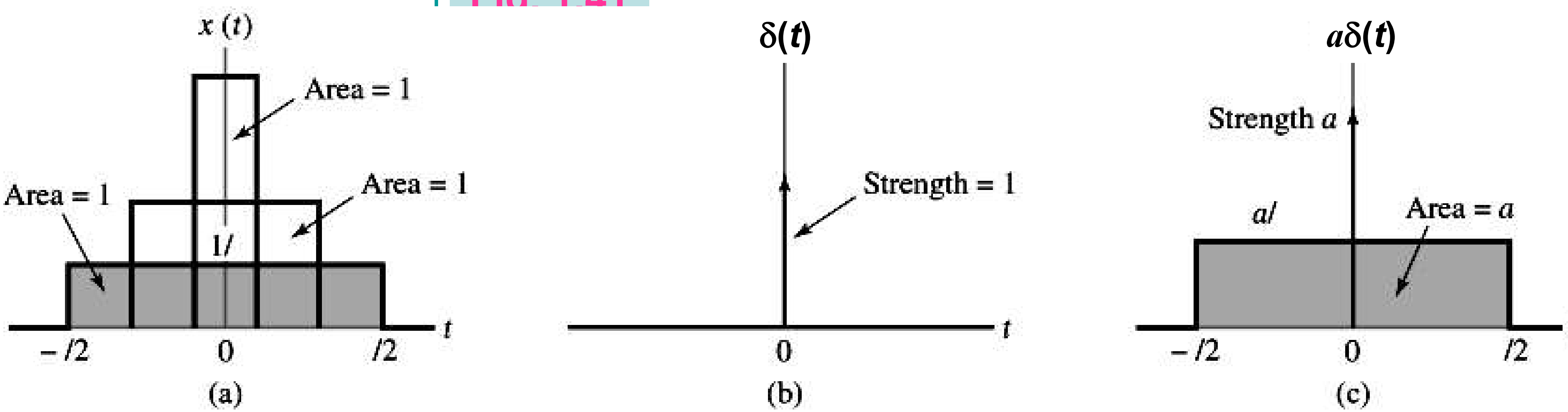

Figure 1.42 (p. 46)
(a) Evolution of a rectangular pulse of unit area into an impulse of unit strength (i.e., unit impulse). (b) Graphical symbol for unit impulse.
(c) Representation of an impulse of strength *a* that results from allowing the duration Δ of a rectangular pulse of area *a* to approach zero.

Introduction

◆ **Continuous-time case:**

Dirac delta function

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0 \qquad (1.59)$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad (1.60)$$

1. As the duration decreases, the rectangular pulse approximates the impulse more closely.

⇨ **Fig. 1.42.**

2. Mathematical relation between impulse and rectangular pulse function:

$$\delta(t) = \lim_{\Delta \to 0} x_{\Delta}(t) \qquad (1.61)$$

1. **$x_{\Delta}(t)$: even function of t, Δ = duration.**

2. **$x_{\Delta}(t)$: Unit area.**

⇨ **Fig. 1.42 (a).**

3. $\delta(t)$ is the derivative of $u(t)$:

[redacted] **(1.62)**

4. $u(t)$ is the integral of $\delta(t)$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad (1.63)$$

Example 1.10 *RC* Circuit (Continued)

For the RC circuit shown in Fig. 1.43 (a), determine the current $i(t)$ that flows through the capacitor for $t \geq 0$.

Introduction

◆ **Properties of impulse function:**

1. Even function: $\delta(-t) = \delta(t)$ **(1.64)**

2. Sifting property:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \quad (1.65)$$

3. Time-scaling property:

$$\delta(at) = \frac{1}{a}\delta(t), \quad a > 0 \quad (1.66)$$

<p.f.> Fig. 1.44

1. Rectangular pulse approximation:

$$\delta(at) = \lim_{\Delta \to 0} x_\Delta(at) \quad (1.67)$$

2. Unit area pulse: Fig. 1.44(a).

Time scaling: Fig. 1.44(b).

Area = 1/a

Restoring unit area ⇨ $ax_\Delta(at)$

⇨ $$\lim_{\Delta \to 0} x_\Delta(at) = \frac{1}{a}\delta(t) \quad (1.68)$$

Ex. RLC circuit driven by impulsive source: Fig. 1.45.

For Fig. 1.45 (a), **the voltage across the capacitor at time $t = 0^+$ is**

$$V_0 = \frac{1}{C}\int_{0^-}^{0^+} I_0\delta(t)dt = \frac{I_0}{C} \quad (1.69)$$

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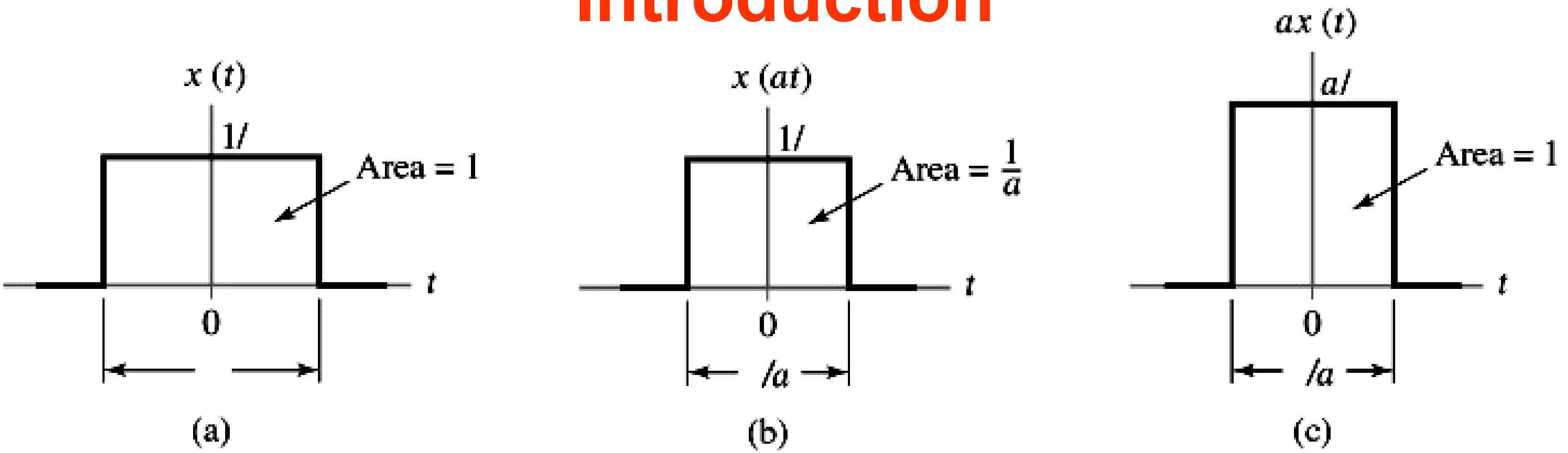
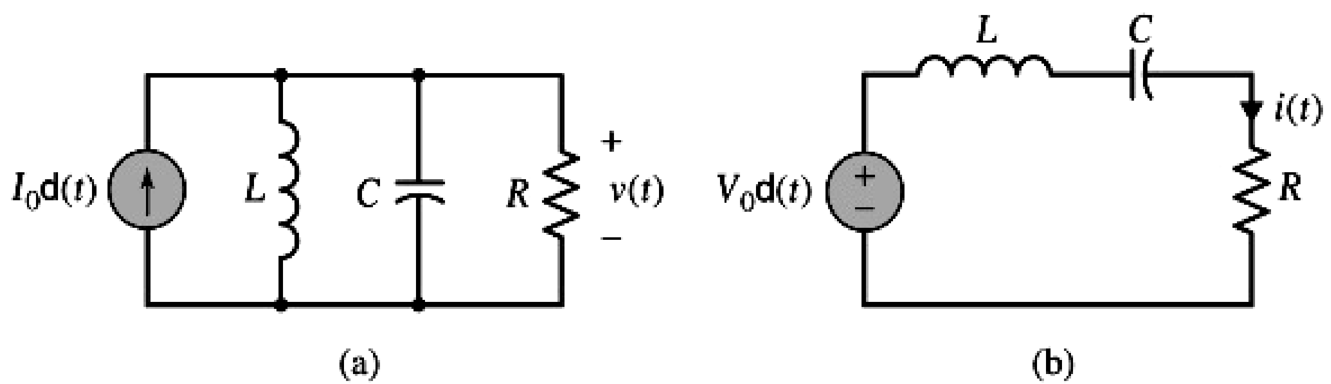


Figure 1.44 (p. 48)
Steps involved in proving the time-scaling property of the unit impulse. (a) Rectangular pulse $x\Delta(t)$ of amplitude $1/\Delta$ and duration Δ, symmetric about the origin. (b) Pulse $x\Delta(t)$ compressed by factor *a*. (c) Amplitude scaling of the compressed pulse, restoring it to unit area.

Figure 1.45 (p. 49)
(a) Parallel *LRC* circuit driven by an impulsive current signal. (b) Series *LRC* circuit driven by an impulsive voltage signal.



Introduction

★ 1.6.7 Derivatives of The Impulse

1. Doublet:

$$\delta^{(1)}(t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left(\delta(t+\Delta/2) - \delta(t-\Delta/2)\right) \quad (1.70)$$

2. Fundamental property of the doublet:

$$\int_{-\infty}^{\infty} \delta^{(1)}(t)dt = 0 \quad (1.71)$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(1)}(t-t_0)dt = \frac{d}{dt} f(t)\Big|_{t=t_0} \quad (1.72)$$

3. Second derivative of impulse:

$$\frac{\partial^2}{\partial t^2}\delta(t) = \frac{d}{dt}\delta^{(1)}(t) = \lim_{\Delta \to 0} \frac{\delta^{(1)}(t+\Delta/2) - \delta^{(1)}(t-\Delta/2)}{\Delta} \quad (1.73)$$

Problem 1.24

$$\int_{-\infty}^{\infty} f(t)\delta^{(2)}(t-t_0)dt = \frac{d^2}{dt^2} f(t)\big|_{t=t_0}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t-t_0)dt = \frac{d^n}{dt^n} f(t)\big|_{t=t_0}$$

★ 1.6.8 Ramp Function

1. Continuous-time case:

$$r(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases} \quad (1.74)$$

or

$$r(t) = tu(t) \quad (1.75)$$

⇨ Fig. 1.46

Introduction

2. Discrete-time case:

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad \textbf{(1.76)}$$

or

$$r[n] = nu[n] \qquad \textbf{(1.77)}$$

⇨ **Fig. 1.47.**

Figure 1.46 (p. 51)
Ramp function of unit slope.

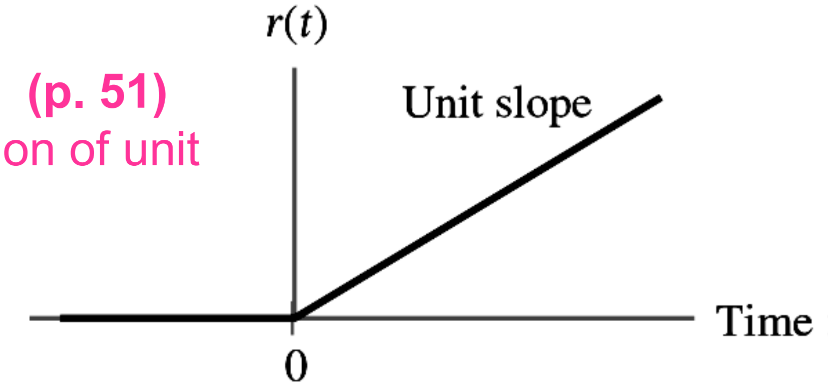
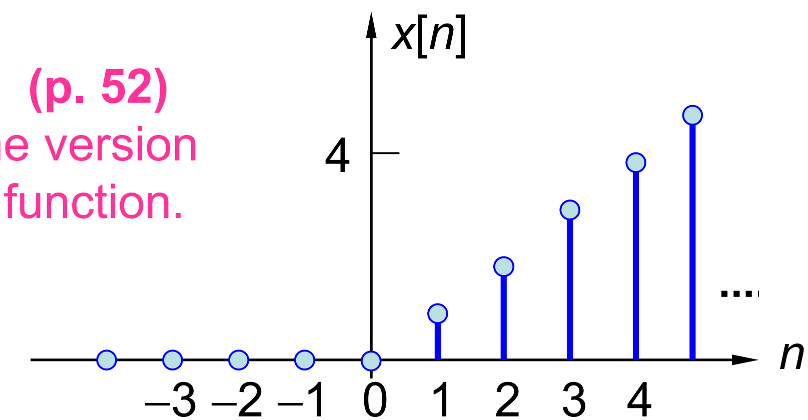


Figure 1.47 (p. 52)
Discrete-time version of the ramp function.



Example 1.11 Parallel Circuit

Consider the parallel circuit of Fig. 1-48 (a) involving a dc current source I_0 and an initially uncharged capacitor *C*.

The switch across the capacitor is suddenly opened at time t = 0. Determine the current $i(t)$ flowing through the capacitor and the voltage $v(t)$ across it for $t \geq 0$.

<Sol.>

1. Capacitor current: $i(t) = I_0 u(t)$

Introduction

2. Capacitor voltage:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} I_0 u(\tau)\,d\tau$$

$$= \begin{cases} 0 & \text{for} \quad t < 0 \\ \dfrac{I_0}{C}t & \text{for} \quad t > 1 \end{cases}$$

$$= \frac{I_0}{C} t u(t)$$

$$= \frac{I_0}{C} r(t)$$

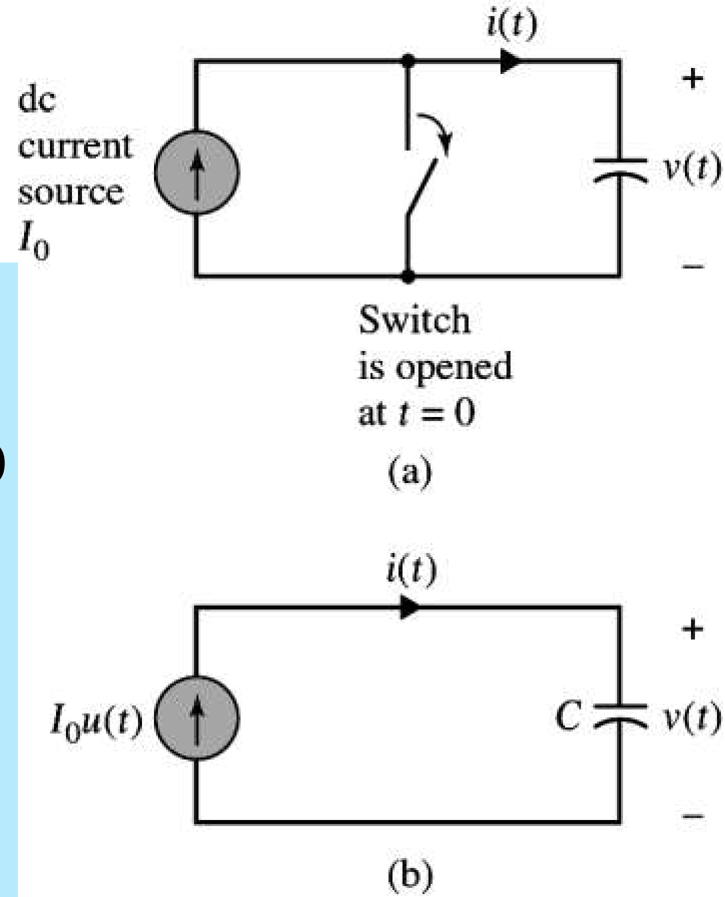


Figure 1.48 (p. 52) (a) Parallel circuit consisting of a current source, switch, and capacitor, the capacitor is initially assumed to be uncharged, and the switch is opened at time $t = 0$. (b) Equivalent circuit replacing the action of opening the switch with the step function $u(t)$.

Introduction

1.7 Systems Viewed as Interconnections of Operations

A system may be viewed as an *interconnection of operations* that transforms an input signal into an output signal with properties different from those of the input signal.

1. Continuous-time case:

$$y(t) = H\{x(t)\} \qquad \textbf{(1.78)}$$

2. Discrete-time case:

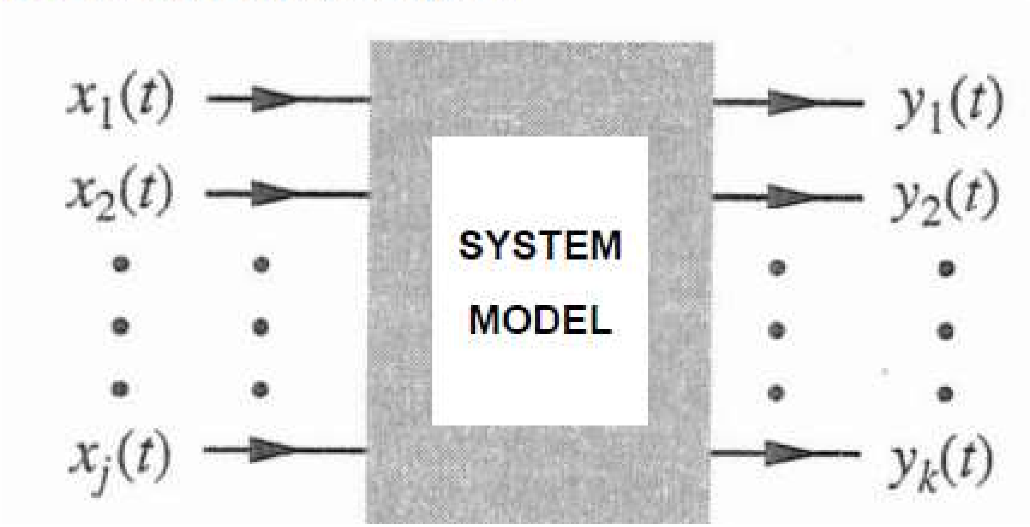
$$y[n] = H\{x[n]\} \qquad \textbf{(1.79)}$$

Fig. 1-49 (a) and (b).

Figure 1.49 (p. 53)
Block diagram representation of operator *H* for (a) continuous time and (b) discrete time.

What are Systems?

- Systems are used to **process signals** to **modify** or **extract information**
- Physical system – characterized by their **input-output relationships**
- E.g. electrical systems are characterized by voltage-current relationships for components and the **laws of interconnections** (i.e. Kirchhoff's laws)
- From this, we derive a **mathematical model** of the system
- "**Black box**" model of a system:



Linear Systems (1)

- A **linear system** exhibits the **additivity** property:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad x_1 + x_2 \longrightarrow y_1 + y_2$$

- It also must satisfy the **homogeneity** or **scaling** property:

$$x \longrightarrow y \qquad kx \longrightarrow ky$$

- These can be combined into the property of **superposition**:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$$

- A non-linear system is one that is NOT linear (i.e. does not obey the principle of superposition)

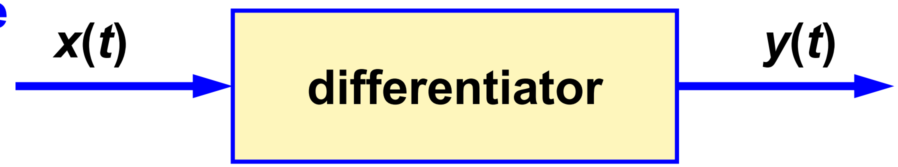
Introduction

1.10 Theme Example

★ 1.10.1 Differentiation and Integration: *RC* Circuits

1. Differentiator ⇒ Sharpening of a pulse

$$y(t) = \frac{d}{dt}x(t) \quad \textbf{(1.99)}$$



1) Simple *RC* circuit: Fig. 1.62.

2) Input-output relation:

[redacted]

$$\Rightarrow \quad \frac{d}{dt}v_2(t) + \frac{1}{RC}v_2(t) = \frac{d}{dt}v_1(t) \quad \textbf{(1.100)}$$

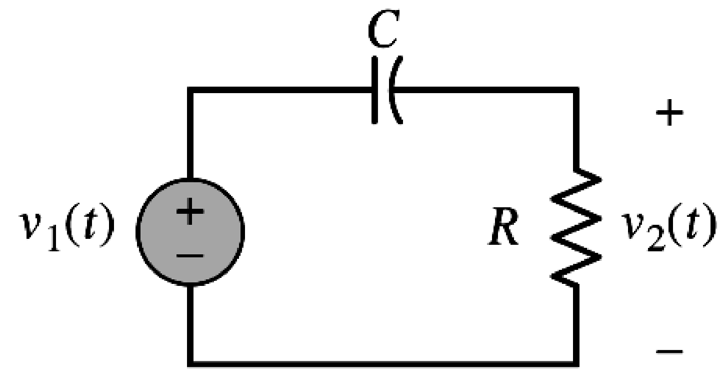


Figure 1.62 (p. 71)
Simple *RC* circuit with small time constant, used as an approximator to a differentiator.

If RC **(time constant) is small enough such that (1.100) is dominated by the second term $v_2(t)/RC$, then**

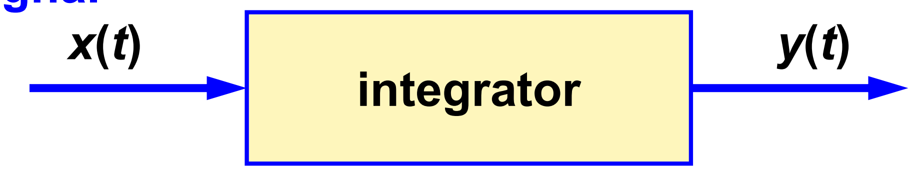
$$\frac{1}{RC}v_2(t) \approx \frac{d}{dt}v_1(t) \quad \Rightarrow \quad v_2(t) \approx RC\frac{d}{dt}v_1(t) \quad \text{for} \quad RC \quad \text{small} \quad \textbf{(1.101)}$$

Introduction

Input: $x(t) = RCv_1(t)$; output: $y(t) = v_2(t)$

2. Integrator ⇒ smoothing of an input signal

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad (1.102)$$



1) Simple *RC* circuit: Fig. 1.63.

2) Input-output relation:

$$RC\frac{d}{dt}v_2(t) + v_2(t) = v_1(t)$$

$$\Rightarrow \quad RCv_2(t) + \int_{-\infty}^{t} v_2(\tau)d\tau = \int_{-\infty}^{t} v_1(\tau)d\tau \qquad (1.103)$$

If RC (time constant) is large enough such that (1.103) is dominated by the first term $RCv_2(t)$, then

$$RCv_2(t) \approx \int_{-\infty}^{t} v_1(\tau)d\tau$$

$$\Rightarrow \quad v_2(t) \approx \frac{1}{RC}\int_{-\infty}^{t} v_1(\tau)d\tau \quad \text{for large } RC$$

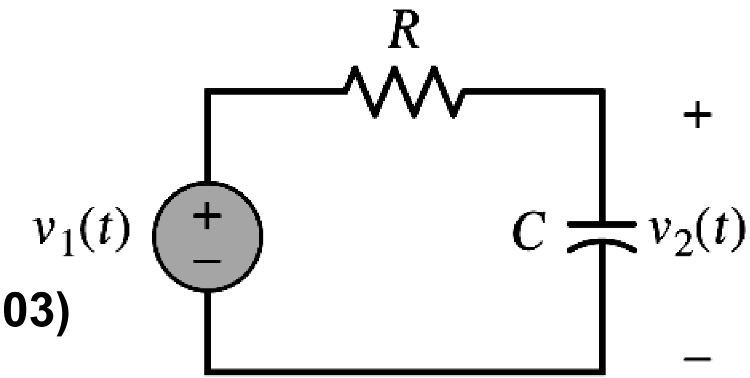


Figure 1.63 (p. 72)
Simple *RC* circuit with large time constant used as an approximator to an integrator.

⇒ Input: $x(t) = [1/(RC)v_1(t)]$; output: $y(t) = v_2(t)$

Introduction

1.9 Noise

Noise $\Rightarrow$ Unwanted signals

1. ***External sources of noise***: atmospheric noise, galactic noise, and human-made noise.
2. ***Internal sources of noise***: spontaneous fluctuations of the current or voltage signal in electrical circuit. (***electrical noise***)

$\Rightarrow$ Fig. 1.60.

★ 1.9.1 Thermal Noise

Thermal noise arises from the random motion of electrons in a conductor.

Two characteristics of thermal noise:

1. Time-averaged value:

$$\bar{v} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(t)\,dt \qquad (1.94)$$

$2T$ = total observation interval of noise

As $T \to \infty$, $\bar{v} \to 0$ Refer to Fig. 1.60.

2. Time-average-squared value:

$$\overline{v^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)\,dt \qquad (1.95)$$

As $T \to \infty$, $\Rightarrow$

$$\overline{v^2} = 4kT_{abs}R\Delta f \quad \text{volts}^2 \qquad (1.96)$$

k = Boltzmann's constant = 1.38×10^{-23} J/K
T_{abs} = absolute temperature

Dr. Hany Ahmed Date: 9/2017

Introduction

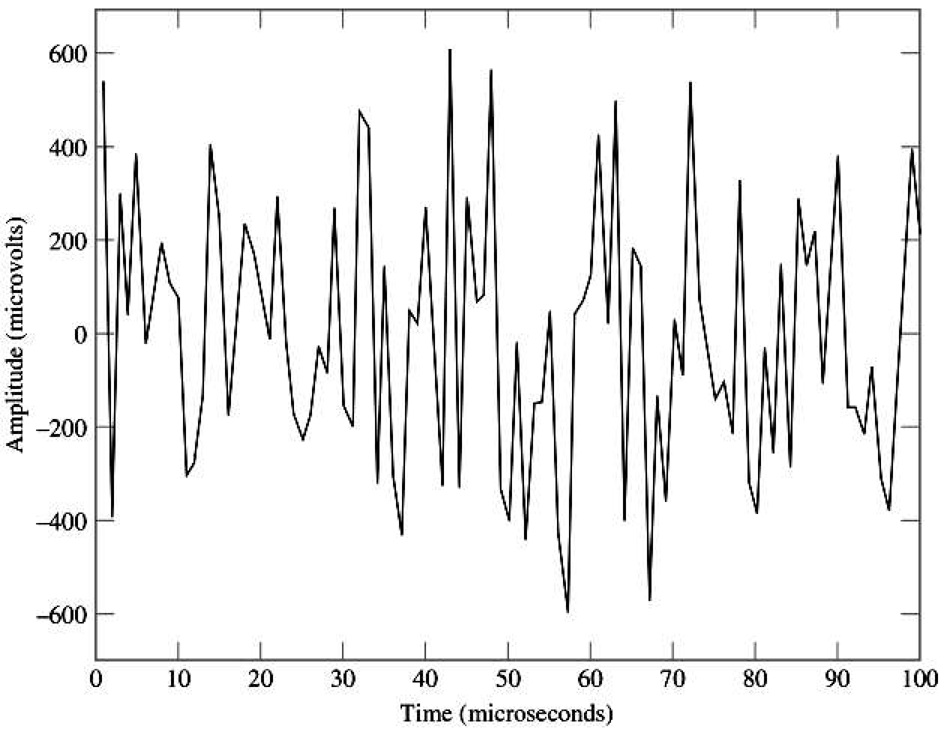


Figure 1.60
(p. 68)
Sample waveform of electrical noise generated by a thermionic diode with a heated cathode. Note that the time-averaged value of the noise voltage displayed is approximately zero.

♣ **Two operating factor that affect available noise power:**

1. **The temperature at which the resistor is maintained.**
2. **The width of the frequency band over which the noise voltage across the resistor is measured.**

★ **1.9.2 Other Sources of Electrical Noise**

1. **Shot noise: the discrete nature of current flow electronic devices**
2. **Ex. Photodetector:**