10 Mechanical Springs

Chapter Outline

When a designer wants rigidity, negligible deflection is an acceptable approximation as long as it does not compromise function. Flexibility is sometimes needed and is often provided by metal bodies with cleverly controlled geometry. These bodies can exhibit flexibility to the degree the designer seeks. Such flexibility can be linear or nonlinear in relating deflection to load. These devices allow controlled application of force or torque; the storing and release of energy can be another purpose. Flexibility allows temporary distortion for access and the immediate restoration of function. Because of machinery's value to designers, springs have been intensively studied; moreover, they are mass-produced (and therefore low cost), and ingenious configurations have been found for a variety of desired applications. In this chapter we will discuss the more frequently used types of springs, their necessary parametric relationships, and their design.

In general, springs may be classified as wire springs, flat springs, or special-shaped springs, and there are variations within these divisions. Wire springs include helical springs of round or square wire, made to resist and deflect under tensile, compressive, or torsional loads. Flat springs include cantilever and elliptical types, wound motor- or clock-type power springs, and flat spring washers, usually called Belleville springs.

10–1 Stresses in Helical Springs

Figure 10–1*a* shows a round-wire helical compression spring loaded by the axial force F. We designate D as the *mean coil diameter* and d as the *wire diameter*. Now imagine that the spring is cut at some point (Fig. 10–1*b*), a portion of it removed, and the effect of the removed portion replaced by the net internal reactions. Then, as shown in the figure, from equilibrium the cut portion would contain a direct shear force F and a torsion $T = FD/2$.

To visualize the torsion, picture a coiled garden hose. Now pull one end of the hose in a straight line perpendicular to the plane of the coil. As each turn of hose is pulled off the coil, the hose twists or turns about its own axis. The flexing of a helical spring creates a torsion in the wire in a similar manner.

The maximum stress in the wire may be computed by superposition of the direct shear stress given by Eq. (3–23), p. 85, and the torsional shear stress given by Eq. (3–37), p. 96. The result is

$$\tau_{\max} = \frac{Tr}{J} + \frac{F}{A} \qquad (a)$$

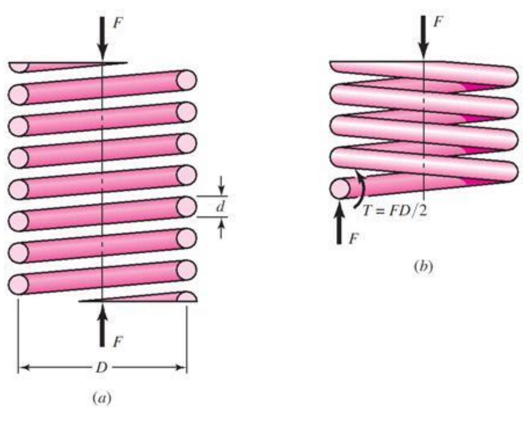

Figure 10–1

(*a*) Axially loaded helical spring; (*b*) free-body diagram showing that the wire is subjected to a direct shear and a torsional shear.

at the *inside* fiber of the spring. Substitution of $\tau_{max} = \tau$, $T = FD/2$, $r = d/2$, $J = \pi d^4/32$, and $A = \pi d^2/4$ gives

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2} \tag{10–1}$$

Now we define the *spring index*

$$C = \frac{D}{d} \tag{10–2}$$

which is a measure of coil curvature. With this relation, Eq. (10–1) can be rearranged to give

$$\tau = K_s \frac{8FD}{\pi d^3} \tag{10–3}$$

where K_s is a *shear-stress correction factor* and is defined by the equation

$$K_s = \frac{2C + 1}{2C} \tag{10–4}$$

For most springs, C ranges from about 6 to 12. Equation (10–3) is quite general and applies for both static and dynamic loads.

The use of square or rectangular wire is not recommended for springs unless space limitations make it necessary. Springs of special wire shapes are not made in large quantities, unlike those of round wire; they have not had the benefit of refining development and hence may not be as strong as springs made from round wire. When space is severely limited, the use of nested round-wire springs should always be considered. They may have an economical advantage over the special-section springs, as well as a strength advantage.

10–2 The Curvature Effect

Equation (10–1) is based on the wire being straight. However, the curvature of the wire increases the stress on the inside of the spring but decreases it only slightly on the outside. This curvature stress is primarily important in fatigue because the loads are lower and there is no opportunity for localized yielding. For static loading, these stresses can normally be neglected because of strain-strengthening with the first application of load.

Unfortunately, it is necessary to find the curvature factor in a roundabout way. The reason for this is that the published equations also include the effect of the direct shear stress. Suppose K_s in Eq. (10–3) is replaced by another K factor, which corrects for both curvature and direct shear. Then this factor is given by either of the equations

$$K_W = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} \tag{10–5}$$

$$K_B = \frac{4C + 2}{4C - 3} \tag{10–6}$$

The first of these is called the *Wahl factor,* and the second, the *Bergsträsser factor.*[1] Since the results of these two equations differ by less than 1 percent, Eq. (10–6) is

[1]Cyril Samónov, "Some Aspects of Design of Helical Compression Springs," *Int. Symp. Design and Synthesis,* Tokyo, 1984.

preferred. The curvature correction factor can now be obtained by canceling out the effect of the direct shear. Thus, using Eq. (10–6) with Eq. (10–4), the curvature correction factor is found to be

$$K_c = \frac{K_B}{K_s} = \frac{2C(4C+2)}{(4C-3)(2C+1)} \tag{10–7}$$

Now, K_s, K_B or K_W, and K_c are simply stress correction factors applied multiplicatively to Tr/J at the critical location to estimate a particular stress. There is *no* stress concentration factor. In this book we will use $\tau = K_B(8FD)/(\pi d^3)$ to predict the largest shear stress.

10–3 Deflection of Helical Springs

The deflection-force relations are quite easily obtained by using Castigliano's theorem. The total strain energy for a helical spring is composed of a torsional component and a shear component. From Eqs. (4–16) and (4–17), p. 156, the strain energy is

$$U = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG} \tag{a}$$

Substituting $T = FD/2$, $l = \pi DN$, $J = \pi d^4/32$, and $A = \pi d^2/4$ results in

$$U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 DN}{d^2 G} \tag{b}$$

where $N = N_a$ = number of active coils. Then using Castigliano's theorem, Eq. (4–20), p. 158, to find total deflection y gives

$$y = \frac{\partial U}{\partial F} = \frac{8FD^3 N}{d^4 G} + \frac{4FDN}{d^2 G} \tag{c}$$

Since $C = D/d$, Eq. (*c*) can be rearranged to yield

$$y = \frac{8FD^3 N}{d^4 G}\left(1 + \frac{1}{2C^2}\right) \doteq \frac{8FD^3 N}{d^4 G} \tag{10–8}$$

The spring rate, also called the *scale* of the spring, is $k = F/y$, and so

$$k \doteq \frac{d^4 G}{8D^3 N} \tag{10–9}$$

10–4 Compression Springs

The four types of ends generally used for compression springs are illustrated in Fig. 10–2. A spring with *plain ends* has a noninterrupted helicoid; the ends are the same as if a long spring had been cut into sections. A spring with plain ends that are *squared* or *closed* is obtained by deforming the ends to a zero-degree helix angle. Springs should always be both squared and ground for important applications, because a better transfer of the load is obtained.

Table 10–1 shows how the type of end used affects the number of coils and the spring length.[2] Note that the digits 0, 1, 2, and 3 appearing in Table 10–1 are often

[2]For a thorough discussion and development of these relations, see Cyril Samónov, "Computer-Aided Design of Helical Compression Springs," ASME paper No. 80-DET-69, 1980.

Figure 10–2

Types of ends for compression springs: (*a*) both ends plain; (*b*) both ends squared; (*c*) both ends squared and ground; (*d*) both ends plain and ground.

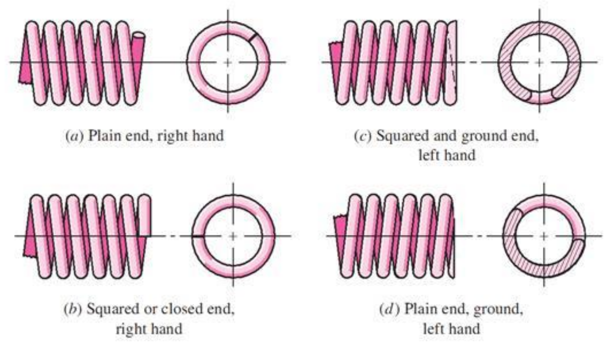

Table 10–1

Formulas for the Dimensional Characteristics of Compression-Springs. (N_a = Number of Active Coils)

Source: From *Design Handbook,* 1987, p. 32. Courtesy of Associated Spring.

	Type of Spring Ends			
Term	**Plain**	**Plain and Ground**	**Squared or Closed**	**Squared and Ground**
End coils, N_e	0	1	2	2
Total coils, N_t	N_a	$N_a + 1$	$N_a + 2$	$N_a + 2$
Free length, L_0	$pN_a + d$	$p(N_a + 1)$	$pN_a + 3d$	$pN_a + 2d$
Solid length, L_s	$d(N_t + 1)$	dN_t	$d(N_t + 1)$	dN_t
Pitch, p	$(L_0 - d)/N_a$	$L_0/(N_a + 1)$	$(L_0 - 3d)/N_a$	$(L_0 - 2d)/N_a$

used without question. *Some of these need closer scrutiny as they may not be integers.* This depends on how a springmaker forms the ends. Forys[3] pointed out that squared and ground ends give a solid length L_s of

$$L_s = (N_t - a)d$$

where a varies, with an average of 0.75, so the entry dN_t in Table 10–1 may be overstated. The way to check these variations is to take springs from a particular springmaker, close them solid, and measure the solid height. Another way is to look at the spring and count the wire diameters in the solid stack.

Set removal or *presetting* is a process used in the manufacture of compression springs to induce useful residual stresses. It is done by making the spring longer than needed and then compressing it to its solid height. This operation *sets* the spring to the required final free length and, since the torsional yield strength has been exceeded, induces residual stresses opposite in direction to those induced in service. Springs to be preset should be designed so that 10 to 30 percent of the initial free length is removed during the operation. If the stress at the solid height is greater than 1.3 times the torsional yield strength, distortion may occur. If this stress is much less than 1.1 times, it is difficult to control the resulting free length.

Set removal increases the strength of the spring and so is especially useful when the spring is used for energy-storage purposes. However, set removal should not be used when springs are subject to fatigue.

[3]Edward L. Forys, "Accurate Spring Heights," *Machine Design,* vol. 56, no. 2, January 26, 1984.

10–5 Stability

In Chap. 4 we learned that a column will buckle when the load becomes too large. Similarly, compression coil springs may buckle when the deflection becomes too large. The critical deflection is given by the equation

$$y_{cr} = L_0 C_1' \left[1 - \left(1 - \frac{C_2'}{\lambda_{eff}^2} \right)^{1/2} \right] \tag{10–10}$$

where y_{cr} is the deflection corresponding to the onset of instability. Samónov[4] states that this equation is cited by Wahl[5] and verified experimentally by Haringx.[6] The quantity λ_{eff} in Eq. (10–10) is the *effective slenderness ratio* and is given by the equation

$$\lambda_{eff} = \frac{\alpha L_0}{D} \tag{10–11}$$

C_1' and C_2' are elastic constants defined by the equations

$$C_1' = \frac{E}{2(E - G)}$$

$$C_2' = \frac{2\pi^2(E - G)}{2G + E}$$

Equation (10–11) contains the *end-condition constant* α. This depends upon how the ends of the spring are supported. Table 10–2 gives values of α for usual end conditions. Note how closely these resemble the end conditions for columns.

Absolute stability occurs when, in Eq. (10–10), the term C_2'/λ_{eff}^2 is greater than unity. This means that the condition for absolute stability is that

$$L_0 < \frac{\pi D}{\alpha} \left[\frac{2(E - G)}{2G + E} \right]^{1/2} \tag{10–12}$$

Table 10–2

End-Condition Constants α for Helical Compression Springs*

End Condition	Constant α
Spring supported between flat parallel surfaces (fixed ends)	0.5
One end supported by flat surface perpendicular to spring axis (fixed); other end pivoted (hinged)	0.707
Both ends pivoted (hinged)	1
One end clamped; other end free	2

*Ends supported by flat surfaces must be squared and ground.

[4]Cyril Samónov "Computer-Aided Design," op. cit.

[5]A. M. Wahl, *Mechanical Springs,* 2d ed., McGraw-Hill, New York, 1963.

[6]J. A. Haringx, "On Highly Compressible Helical Springs and Rubber Rods and Their Application for Vibration-Free Mountings," I and II, *Philips Res. Rep.*, vol. 3, December 1948, pp. 401–449, and vol. 4, February 1949, pp. 49–80

For steels, this turns out to be

$$L_0 < 2.63\frac{D}{\alpha} \tag{10–13}$$

For squared and ground ends $\alpha = 0.5$ and $L_0 < 5.26D$.

10–6 Spring Materials

Springs are manufactured either by hot- or cold-working processes, depending upon the size of the material, the spring index, and the properties desired. In general, prehardened wire should not be used if $D/d < 4$ or if $d > \frac{1}{4}$ in. Winding of the spring induces residual stresses through bending, but these are normal to the direction of the torsional working stresses in a coil spring. Quite frequently in spring manufacture, they are relieved, after winding, by a mild thermal treatment.

A great variety of spring materials are available to the designer, including plain carbon steels, alloy steels, and corrosion-resisting steels, as well as nonferrous materials such as phosphor bronze, spring brass, beryllium copper, and various nickel alloys. Descriptions of the most commonly used steels will be found in Table 10–3. The UNS steels listed in Appendix A should be used in designing hot-worked, heavy-coil springs, as well as flat springs, leaf springs, and torsion bars.

Spring materials may be compared by an examination of their tensile strengths; these vary so much with wire size that they cannot be specified until the wire size is known. The material and its processing also, of course, have an effect on tensile strength. It turns out that the graph of tensile strength versus wire diameter is almost a straight line for some materials when plotted on log-log paper. Writing the equation of this line as

$$S_{ut} = \frac{A}{d^m} \tag{10–14}$$

furnishes a good means of estimating minimum tensile strengths when the intercept A and the slope m of the line are known. Values of these constants have been worked out from recent data and are given for strengths in units of kpsi and MPa in Table 10–4. In Eq. (10–14) when d is measured in millimeters, then A is in $\text{MPa}\cdot\text{mm}^m$ and when d is measured in inches, then A is in $\text{kpsi}\cdot\text{in}^m$.

Although the torsional yield strength is needed to design the spring and to analyze the performance, spring materials customarily are tested only for tensile strength—perhaps because it is such an easy and economical test to make. A very rough estimate of the torsional yield strength can be obtained by assuming that the tensile yield strength is between 60 and 90 percent of the tensile strength. Then the distortion-energy theory can be employed to obtain the torsional yield strength ($S_{ys} = 0.577S_y$). This approach results in the range

$$0.35S_{ut} \le S_{sy} \le 0.52S_{ut} \tag{10–15}$$

for steels.

For wires listed in Table 10–5, the maximum allowable shear stress in a spring can be seen in column 3. Music wire and hard-drawn steel spring wire have a low end of range $S_{sy} = 0.45S_{ut}$. Valve spring wire, Cr-Va, Cr-Si, and other (not shown) hardened and tempered carbon and low-alloy steel wires as a group have $S_{sy} \ge 0.50S_{ut}$. Many nonferrous materials (not shown) as a group have $S_{sy} \ge 0.35S_{ut}$. In view of this,

Table 10–3

High-Carbon and Alloy Spring Steels

Source: From Harold C. R. Carlson, "Selection and Application of Spring Materials," *Mechanical Engineering*, vol. 78, 1956, pp. 331–334.

Name of Material	Similar Specifications	Description
Music wire, 0.80–0.95*C*	UNS G10850 AISI 1085 ASTM A228-51	This is the best, toughest, and most widely used of all spring materials for small springs. It has the highest tensile strength and can withstand higher stresses under repeated loading than any other spring material. Available in diameters 0.12 to 3 mm (0.005 to 0.125 in). Do not use above 120°C (250°F) or at subzero temperatures.
Oil-tempered wire, 0.60–0.70*C*	UNS G10650 AISI 1065 ASTM 229-41	This general-purpose spring steel is used for many types of coil springs where the cost of music wire is prohibitive and in sizes larger than available in music wire. Not for shock or impact loading. Available in diameters 3 to 12 mm (0.125 to 0.5000 in), but larger and smaller sizes may be obtained. Not for use above 180°C (350°F) or at subzero temperatures.
Hard-drawn wire, 0.60–0.70*C*	UNS G10660 AISI 1066 ASTM A227-47	This is the cheapest general-purpose spring steel and should be used only where life, accuracy, and deflection are not too important. Available in diameters 0.8 to 12 mm (0.031 to 0.500 in). Not for use above 120°C (250°F) or at subzero temperatures.
Chrome-vanadium	UNS G61500 AISI 6150 ASTM 231-41	This is the most popular alloy spring steel for conditions involving higher stresses than can be used with the high-carbon steels and for use where fatigue resistance and long endurance are needed. Also good for shock and impact loads. Widely used for aircraft-engine valve springs and for temperatures to 220°C (425°F). Available in annealed or pretempered sizes 0.8 to 12 mm (0.031 to 0.500 in) in diameter.
Chrome-silicon	UNS G92540 AISI 9254	This alloy is an excellent material for highly stressed springs that require long life and are subjected to shock loading. Rockwell hardnesses of C50 to C53 are quite common, and the material may be used up to 250°C (475°F). Available from 0.8 to 12 mm (0.031 to 0.500 in) in diameter.

Table 10–4

Constants A and m of $S_{ut} = A/d^m$ for Estimating Minimum Tensile Strength of Common Spring Wires

Source: From *Design Handbook*, 1987, p. 19. Courtesy of Associated Spring.

Material	ASTM No.	Exponent m	Diameter, in	A, kpsi · in^m	Diameter, mm	A, MPa · mm^m	Relative Cost of wire
Music wire*	A228	0.145	0.004–0.256	201	0.10–6.5	2211	2.6
OQ&T wire†	A229	0.187	0.020–0.500	147	0.5–12.7	1855	1.3
Hard-drawn wire‡	A227	0.190	0.028–0.500	140	0.7–12.7	1783	1.0
Chrome-vanadium wire§	A232	0.168	0.032–0.437	169	0.8–11.1	2005	3.1
Chrome-silicon wire‖	A401	0.108	0.063–0.375	202	1.6–9.5	1974	4.0
302 Stainless wire#	A313	0.146	0.013–0.10	169	0.3–2.5	1867	7.6–11
		0.263	0.10–0.20	128	2.5–5	2065	
		0.478	0.20–0.40	90	5–10	2911	
Phosphor-bronze wire**	B159	0	0.004–0.022	145	0.1–0.6	1000	8.0
		0.028	0.022–0.075	121	0.6–2	913	
		0.064	0.075–0.30	110	2–7.5	932	

*Surface is smooth, free of defects, and has a bright, lustrous finish.

†Has a slight heat-treating scale which must be removed before plating.

‡Surface is smooth and bright with no visible marks.

§Aircraft-quality tempered wire, can also be obtained annealed.

‖Tempered to Rockwell C49, but may be obtained untempered.

#Type 302 stainless steel.

**Temper CA510.

Joerres[7] uses the maximum allowable torsional stress for static application shown in Table 10–6. For specific materials for which you have torsional yield information use this table as a guide. Joerres provides set-removal information in Table 10–6, that $S_{sy} \geq 0.65 S_{ut}$ increases strength through cold work, but at the cost of an additional operation by the springmaker. Sometimes the additional operation can be done by the manufacturer during assembly. Some correlations with carbon steel springs show that the tensile yield strength of spring wire in torsion can be estimated from $0.75 S_{ut}$. The corresponding estimate of the yield strength in shear based on distortion energy theory is $S_{sy} = 0.577(0.75) S_{ut} = 0.433 S_{ut} \doteq 0.45 S_{ut}$. Samónov discusses the problem of allowable stress and shows that

$$S_{sy} = \tau_{\text{all}} = 0.56 S_{ut} \tag{10–16}$$

for high-tensile spring steels, which is close to the value given by Joerres for hardened alloy steels. He points out that this value of allowable stress is specified by Draft Standard 2089 of the German Federal Republic when Eq. (10–3) is used without stress-correction factor.

[7]Robert E. Joerres, "Springs," Chap. 6 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design*, 3rd ed., McGraw-Hill, New York, 2004.

Table 10–5

Mechanical Properties of Some Spring Wires

Material	Elastic Limit, Percent of S_{ut} Tension	Torsion	Diameter d, in	E Mpsi	E GPa	G Mpsi	G GPa
Music wire A228	65–75	45–60	<0.032	29.5	203.4	12.0	82.7
			0.033–0.063	29.0	200	11.85	81.7
			0.064–0.125	28.5	196.5	11.75	81.0
			>0.125	28.0	193	11.6	80.0
HD spring A227	60–70	45–55	<0.032	28.8	198.6	11.7	80.7
			0.033–0.063	28.7	197.9	11.6	80.0
			0.064–0.125	28.6	197.2	11.5	79.3
			>0.125	28.5	196.5	11.4	78.6
Oil tempered A239	85–90	45–50		28.5	196.5	11.2	77.2
Valve spring A230	85–90	50–60		29.5	203.4	11.2	77.2
Chrome-vanadium A231	88–93	65–75		29.5	203.4	11.2	77.2
A232	88–93			29.5	203.4	11.2	77.2
Chrome-silicon A401	85–93	65–75		29.5	203.4	11.2	77.2
Stainless steel							
A313*	65–75	45–55		28	193	10	69.0
17-7PH	75–80	55–60		29.5	208.4	11	75.8
414	65–70	42–55		29	200	11.2	77.2
420	65–75	45–55		29	200	11.2	77.2
431	72–76	50–55		30	206	11.5	79.3
Phosphor-bronze B159	75–80	45–50		15	103.4	6	41.4
Beryllium-copper B197	70	50		17	117.2	6.5	44.8
	75	50–55		19	131	7.3	50.3
Inconel alloy X-750	65–70	40–45		31	213.7	11.2	77.2

*Also includes 302, 304, and 316.

Note: See Table 10–6 for allowable torsional stress design values.

Table 10–6

Maximum Allowable Torsional Stresses for Helical Compression Springs in Static Applications

Source: Robert E. Joerres, "Springs," Chap. 6 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design*, 3rd ed., McGraw-Hill, New York, 2004.

Material	Maximum Percent of Tensile Strength: Before Set Removed (includes K_W or K_B)	After Set Removed (includes K_s)
Music wire and cold-drawn carbon steel	45	60–70
Hardened and tempered carbon and low-alloy steel	50	65–75
Austenitic stainless steels	35	55–65
Nonferrous alloys	35	55–65

EXAMPLE 10–1

A helical compression spring is made of no. 16 music wire. The outside diameter of the spring is $\frac{7}{16}$ in. The ends are squared and there are $12\frac{1}{2}$ total turns.
(*a*) Estimate the torsional yield strength of the wire.
(*b*) Estimate the static load corresponding to the yield strength.
(*c*) Estimate the scale of the spring.
(*d*) Estimate the deflection that would be caused by the load in part (*b*).
(*e*) Estimate the solid length of the spring.
(*f*) What length should the spring be to ensure that when it is compressed solid and then released, there will be no permanent change in the free length?
(*g*) Given the length found in part (*f*), is buckling a possibility?
(*h*) What is the pitch of the body coil?

Solution

(*a*) From Table A–28, the wire diameter is $d = 0.037$ in. From Table 10–4, we find $A = 201$ kpsi · in^m and $m = 0.145$. Therefore, from Eq. (10–14)

$$S_{ut} = \frac{A}{d^m} = \frac{201}{0.037^{0.145}} = 324 \text{ kpsi}$$

Then, from Table 10–6,

Answer

$$S_{sy} = 0.45 S_{ut} = 0.45(324) = 146 \text{ kpsi}$$

(*b*) The mean spring coil diameter is $D = \frac{7}{16} - 0.037 = 0.400$ in, and so the spring index is $C = 0.400/0.037 = 10.8$. Then, from Eq. (10–6),

$$K_B = \frac{4C+2}{4C-3} = \frac{4(10.8)+2}{4(10.8)-3} = 1.124$$

Now rearrange Eq. (10–3) replacing K_s and τ with K_B and S_{ys}, respectively, and solve for F:

Answer

$$F = \frac{\pi d^3 S_{sy}}{8K_B D} = \frac{\pi(0.037^3)146(10^3)}{8(1.124)\,0.400} = 6.46 \text{ lbf}$$

(*c*) From Table 10–1, $N_a = 12.5 - 2 = 10.5$ turns. In Table 10–5, $G = 11.85$ Mpsi, and the scale of the spring is found to be, from Eq. (10–9),

Answer

$$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.037^4\,(11.85)10^6}{8(0.400^3)10.5} = 4.13 \text{ lbf/in}$$

Answer (*d*)

$$y = \frac{F}{k} = \frac{6.46}{4.13} = 1.56 \text{ in}$$

(*e*) From Table 10–1,

Answer

$$L_s = (N_t + 1)d = (12.5 + 1)0.037 = 0.500 \text{ in}$$

Answer (*f*)

$$L_0 = y + L_s = 1.56 + 0.500 = 2.06 \text{ in.}$$

(*g*) To avoid buckling, Eq. (10–13) and Table 10–2 give

$$L_0 < 2.63\frac{D}{\alpha} = 2.63\frac{0.400}{0.5} = 2.10 \text{ in}$$

Mathematically, a free length of 2.06 in is less than 2.10 in, and buckling is unlikely. However, the forming of the ends will control how close α is to 0.5. This has to be investigated and an inside rod or exterior tube or hole may be needed.

(*h*) Finally, from Table 10–1, the pitch of the body coil is

Answer

$$p = \frac{L_0 - 3d}{N_a} = \frac{2.06 - 3(0.037)}{10.5} = 0.186 \text{ in}$$

10–7 Helical Compression Spring Design for Static Service

The preferred range of spring index is $4 \le C \le 12$, with the lower indexes being more difficult to form (because of the danger of surface cracking) and springs with higher indexes tending to tangle often enough to require individual packing. This can be the first item of the design assessment. The recommended range of active turns is $3 \le N_a \le 15$. To maintain linearity when a spring is about to close, it is necessary to avoid the gradual touching of coils (due to nonperfect pitch). A helical coil spring force-deflection characteristic is ideally linear. Practically, it is nearly so, but not at each end of the force-deflection curve. The spring force is not reproducible for very small deflections, and near closure, nonlinear behavior begins as the number of active turns diminishes as coils begin to touch. The designer confines the spring's operating point to the central 75 percent of the curve between no load, $F = 0$, and closure, $F = F_s$. Thus, the maximum operating force should be limited to $F_{max} \le \frac{7}{8} F_s$. Defining the fractional overrun to closure as ξ, where

$$F_s = (1 + \xi) F_{max} \tag{10–17}$$

it follows that

$$F_s = (1 + \xi) F_{max} = (1 + \xi)\left(\frac{7}{8}\right) F_s$$

From the outer equality $\xi = 1/7 = 0.143 \doteq 0.15$. Thus, it is recommended that $\xi \ge 0.15$.

In addition to the relationships and material properties for springs, we now have some recommended design conditions to follow, namely:

$$4 \le C \le 12 \tag{10–18}$$

$$3 \le N_a \le 15 \tag{10–19}$$

$$\xi \ge 0.15 \tag{10–20}$$

$$n_s \ge 1.2 \tag{10–21}$$

where n_s is the factor of safety at closure (solid height).

When considering designing a spring for high volume production, the figure of merit can be the cost of the wire from which the spring is wound. The fom would be proportional to the relative material cost, weight density, and volume:

$$\text{fom} = -(\text{relative material cost}) \frac{\gamma \pi^2 d^2 N_t D}{4} \tag{10–22}$$

For comparisons between steels, the specific weight γ can be omitted.

Spring design is an open-ended process. There are many decisions to be made, and many possible solution paths as well as solutions. In the past, charts, nomographs,

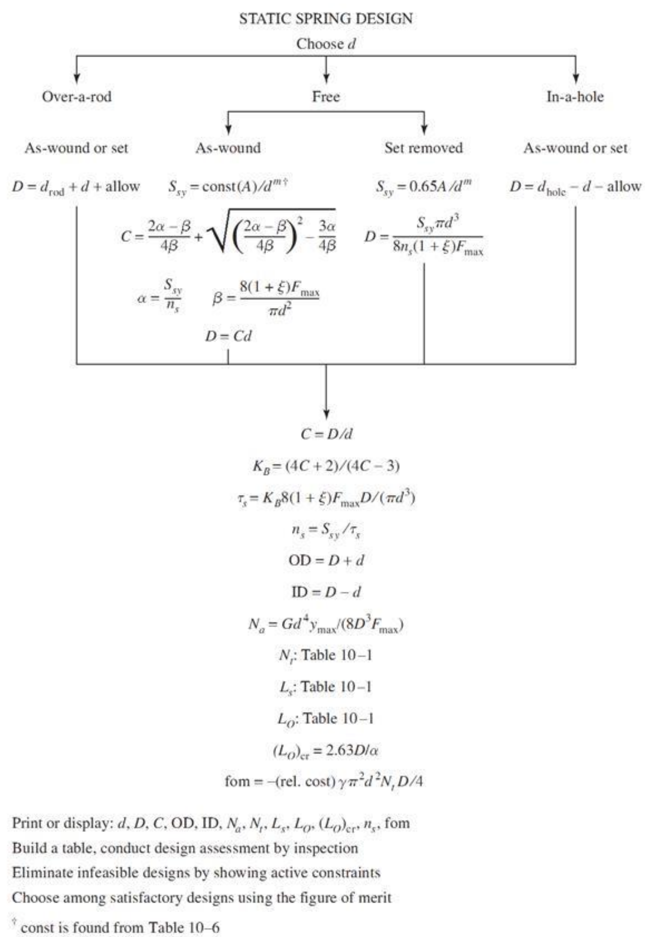

Figure 10–3

Helical coil compression spring design flowchart for static loading.

and "spring design slide rules" were used by many to simplify the spring design problem. Today, the computer enables the designer to create programs in many different formats—direct programming, spreadsheet, MATLAB, etc. Commercial programs are also available.[8] There are almost as many ways to create a spring-design program as there are programmers. Here, we will suggest one possible design approach.

Design Strategy

Make the a priori decisions, with hard-drawn steel wire the first choice (relative material cost is 1.0). Choose a wire size d. With all decisions made, generate a column of parameters: d, D, C, OD or ID, N_a, L_s, L_0, $(L_0)_{cr}$, n_s, and fom. By incrementing wire sizes available, we can scan the table of parameters and apply the design recommendations by inspection. After wire sizes are eliminated, choose the spring design with the highest figure of merit. This will give the optimal design despite the presence

[8]For example, see *Advanced Spring Design,* a program developed jointly between the Spring Manufacturers Institute (SMI), www.smihq.org, and Universal Technical Systems, Inc. (UTS), www.uts.com.

of a discrete design variable d and aggregation of equality and inequality constraints. The column vector of information can be generated by using the flowchart displayed in Fig. 10–3. It is general enough to accommodate to the situations of as-wound and set-removed springs, operating over a rod, or in a hole free of rod or hole. In as-wound springs the controlling equation must be solved for the spring index as follows. From Eq. (10–3) with $\tau = S_{sy}/n_s$, $C = D/d$, K_B from Eq. (10–6), and Eq. (10–17),

$$\frac{S_{sy}}{n_s} = K_B \frac{8F_s D}{\pi d^3} = \frac{4C+2}{4C-3}\left[\frac{8(1+\xi)\,F_{\max} C}{\pi d^2}\right] \tag{a}$$

Let

$$\alpha = \frac{S_{sy}}{n_s} \tag{b}$$

$$\beta = \frac{8\,(1+\xi)\,F_{\max}}{\pi d^2} \tag{c}$$

Substituting Eqs. (*b*) and (*c*) into (*a*) and simplifying yields a quadratic equation in C. The larger of the two solutions will yield the spring index

$$C = \frac{2\alpha-\beta}{4\beta} + \sqrt{\left(\frac{2\alpha-\beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}} \tag{10–23}$$

EXAMPLE 10–2

A music wire helical compression spring is needed to support a 20-lbf load after being compressed 2 in. Because of assembly considerations the solid height cannot exceed 1 in and the free length cannot be more than 4 in. Design the spring.

Solution

The a priori decisions are

- Music wire, A228; from Table 10–4, $A = 201\,000$ psi-in^m; $m = 0.145$; from Table 10–5, $E = 28.5$ Mpsi, $G = 11.75$ Mpsi (expecting $d > 0.064$ in)
- Ends squared and ground
- Function: $F_{\max} = 20$ lbf, $y_{\max} = 2$ in
- Safety: use design factor at solid height of $(n_s)_d = 1.2$
- Robust linearity: $\xi = 0.15$
- Use as-wound spring (cheaper), $S_{sy} = 0.45 S_{ut}$ from Table 10–6
- Decision variable: $d = 0.080$ in, music wire gage #30, Table A–28. From Fig. 10–3 and Table 10–6,

$$S_{sy} = 0.45\frac{201\,000}{0.080^{0.145}} = 130\,455 \text{ psi}$$

From Fig. 10–3 or Eq. (10–23)

$$\alpha = \frac{S_{sy}}{n_s} = \frac{130\,455}{1.2} = 108\,713 \text{ psi}$$

$$\beta = \frac{8(1+\xi)F_{\max}}{\pi d^2} = \frac{8(1+0.15)20}{\pi(0.080^2)} = 9151.4 \text{ psi}$$

$$C = \frac{2(108\,713) - 9151.4}{4(9151.4)} + \sqrt{\left[\frac{2(108\,713) - 9151.4}{4(9151.4)}\right]^2 - \frac{3(108\,713)}{4(9151.4)}} = 10.53$$

Continuing with Fig. 10–3:

$$D = Cd = 10.53(0.080) = 0.8424$$

$$K_B = \frac{4(10.53) + 2}{4(10.53) - 3} = 1.128$$

$$\tau_s = 1.128\frac{8(1 + 0.15)20(0.8424)}{\pi(0.080)^3} = 108\,700 \text{ psi}$$

$$n_s = \frac{130\,445}{108\,700} = 1.2$$

$$\text{OD} = 0.843 + 0.080 = 0.923 \text{ in}$$

$$N_a = \frac{0.080^4(11.75)10^6(2)}{8(0.843)^3 20} = 10.05 \text{ turns}$$

$$N_t = 10.05 + 2 = 12.05 \text{ total turns}$$

$$L_s = 0.080(12.05) = 0.964 \text{ in}$$

$$L_0 = 0.964 + (1 + 0.15)2 = 3.264 \text{ in}$$

$$(L)_{\text{cr}} = 2.63(0.843/0.5) = 4.43 \text{ in}$$

$$\text{fom} = -2.6\pi^2(0.080)^2 12.05(0.843)/4 = -0.417$$

Repeat the above for other wire diameters and form a table (easily accomplished with a spreadsheet program):

d:	0.063	0.067	0.071	0.075	0.080	0.085	0.090	0.095
D	0.391	0.479	0.578	0.688	0.843	1.017	1.211	1.427
C	6.205	7.153	8.143	9.178	10.53	11.96	13.46	15.02
OD	0.454	0.546	0.649	0.763	0.923	1.102	1.301	1.522
N_a	39.1	26.9	19.3	14.2	10.1	7.3	5.4	4.1
L_s	2.587	1.936	1.513	1.219	0.964	0.790	0.668	0.581
L_0	4.887	4.236	3.813	3.519	3.264	3.090	2.968	2.881
$(L_0)_{cr}$	2.06	2.52	3.04	3.62	4.43	5.35	6.37	7.51
n_s	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
fom	−0.409	−0.399	−0.398	−0.404	−0.417	−0.438	−0.467	−0.505

Now examine the table and perform the adequacy assessment. The constraint $3 \le N_a \le 15$ rules out wire diameters less than 0.075 in. The spring index constraint $4 \le C \le 12$ rules out diameters larger than 0.085 in. The $L_s \le 1$ constraint rules out diameters less than 0.080 in. The $L_0 \le 4$ constraint rules out diameters less than 0.071 in. The buckling criterion rules out free lengths longer than $(L_0)_{\text{cr}}$, which rules out diameters less than 0.075 in. The factor of safety n_s is exactly 1.20 because the mathematics forced it. Had the spring been in a hole or over a rod, the helix diameter would be chosen without reference to $(n_s)_d$. The result is that there are only two springs in the feasible domain, one with a wire diameter of 0.080 in and the other with a wire diameter of 0.085. The figure of merit decides and the decision is the design with 0.080 in wire diameter.

Having designed a spring, will we have it made to our specifications? Not necessarily. There are vendors who stock literally thousands of music wire compression springs. By browsing their catalogs, we will usually find several that are close. Maximum deflection and maximum load are listed in the display of characteristics. Check to see if this allows soliding without damage. Often it does not. Spring rates may only be close. At the very least this situation allows a small number of springs to be ordered "off the shelf" for testing. The decision often hinges on the economics of special order versus the acceptability of a close match.

EXAMPLE 10–3 Indexing is used in machine operations when a circular part being manufactured must be divided into a certain number of segments. Figure 10–4 shows a portion of an indexing fixture used to successively position a part for the operation. When the knob is momentarily pulled up, part 6, which holds the workpiece, is rotated about a vertical axis to the next position and locked in place by releasing the index pin. In this example we wish to design the spring to exert a force of about 3 lbf and to fit in the space defined in the figure caption.

Solution Since the fixture is not a high-production item, a stock spring will be selected. These are available in music wire. In one catalog there are 76 stock springs available having an outside diameter of 0.480 in and designed to work in a $\frac{1}{2}$-in hole. These are made in seven different wire sizes, ranging from 0.038 up to 0.063 in, and in free lengths from $\frac{1}{2}$ to $2\frac{1}{2}$ in, depending upon the wire size.

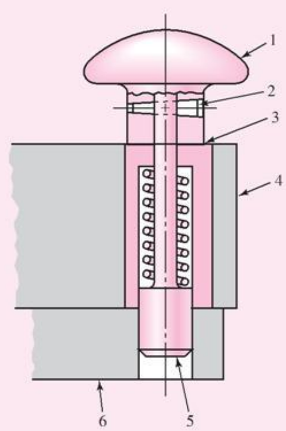

Figure 10–4

Part 1, pull knob; part 2, tapered retaining pin; part 3, hardened bushing with press fit; part 4, body of fixture; part 5, indexing pin; part 6, workpiece holder. Space of the spring is $\frac{5}{8}$ in OD, $\frac{1}{4}$ in ID, and $1\frac{3}{8}$ in long, with the pin down as shown. The pull knob must be raised $\frac{3}{4}$ in to permit indexing.

Since the pull knob must be raised $\frac{3}{4}$ in for indexing and the space for the spring is $1\frac{3}{8}$ in long when the pin is down, the solid length cannot be more than $\frac{5}{8}$ in.

Let us begin by selecting a spring having an outside diameter of 0.480 in, a wire size of 0.051 in, a free length of $1\frac{3}{4}$ in, $11\frac{1}{2}$ total turns, and plain ends. Then $m = 0.145$ and $A = 201$ kpsi · in^m for music wire. Then

$$S_{sy} = 0.45\frac{A}{d^m} = 0.45\frac{201}{0.051^{0.145}} = 139.3 \text{ kpsi}$$

With plain ends, from Table 10–1, the number of active turns is

$$N_a = N_t = 11.5 \text{ turns}$$

The mean coil diameter is $D = \text{OD} - d = 0.480 - 0.051 = 0.429$ in. From Eq. (10–9) the spring rate is, for $G = 11.85(10^6)$ psi from Table 10–5,

$$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.051^4 (11.85)10^6}{8(0.429)^3 11.5} = 11.0 \text{ lbf/in}$$

From Table 10–1, the solid height L_s is

$$L_s = d(N_t + 1) = 0.051(11.5 + 1) = 0.638 \text{ in}$$

The spring force when the pin is down, $F_{\min}$, is

$$F_{\min} = k y_{\min} = 11.0(1.75 - 1.375) = 4.13 \text{ lbf}$$

When the spring is compressed solid, the spring force F_s is

$$F_s = k y_s = k(L_0 - L_s) = 11.0(1.75 - 0.638) = 12.2 \text{ lbf}$$

Since the spring index is $C = D/d = 0.429/0.051 = 8.41$,

$$K_B = \frac{4C + 2}{4C - 3} = \frac{4(8.41) + 2}{4(8.41) - 3} = 1.163$$

and for the as-wound spring, the shear stress when compressed solid is

$$\tau_s = K_B \frac{8F_s D}{\pi d^3} = 1.163\frac{8(12.2)0.429}{\pi (0.051)^3} = 116\,850 \text{ psi}$$

The factor of safety when the spring is compressed solid is

$$n_s = \frac{S_{sy}}{\tau_s} = \frac{139.3}{116.9} = 1.19$$

Since n_s is marginally adequate and L_s is larger than $\frac{5}{8}$ in, we must investigate other springs with a smaller wire size. After several investigations another spring has possibilities. It is as-wound music wire, $d = 0.045$ in, 20 gauge (see Table A–25) $\text{OD} = 0.480$ in, $N_t = 11.5$ turns, $L_0 = 1.75$ in. S_{sy} is still 139.3 kpsi, and

$$D = \text{OD} - d = 0.480 - 0.045 = 0.435 \text{ in}$$

$$N_a = N_t = 11.5 \text{ turns}$$

$$k = \frac{0.045^4 (11.85)10^6}{8(0.435)^3 11.5} = 6.42 \text{ lbf/in}$$

$$L_s = d(N_t + 1) = 0.045(11.5 + 1) = 0.563 \text{ in}$$

$$F_{\min} = ky_{\min} = 6.42(1.75 - 1.375) = 2.41 \text{ lbf}$$

$$F_s = 6.42(1.75 - 0.563) = 7.62 \text{ lbf}$$

$$C = \frac{D}{d} = \frac{0.435}{0.045} = 9.67$$

$$K_B = \frac{4(9.67) + 2}{4(9.67) - 3} = 1.140$$

$$\tau_s = 1.140 \frac{8(7.62)0.435}{\pi(0.045)^3} = 105\,600 \text{ psi}$$

$$n_s = \frac{S_{sy}}{\tau_s} = \frac{139.3}{105.6} = 1.32$$

Now $n_s > 1.2$, buckling is not possible as the coils are guarded by the hole surface, and the solid length is less than $\frac{5}{8}$ in, so this spring is selected. By using a stock spring, we take advantage of economy of scale.

10–8 Critical Frequency of Helical Springs

If a wave is created by a disturbance at one end of a swimming pool, this wave will travel down the length of the pool, be reflected back at the far end, and continue in this back-and-forth motion until it is finally damped out. The same effect occurs in helical springs, and it is called *spring surge*. If one end of a compression spring is held against a flat surface and the other end is disturbed, a compression wave is created that travels back and forth from one end to the other exactly like the swimming-pool wave.

Spring manufacturers have taken slow-motion movies of automotive valve-spring surge. These pictures show a very violent surging, with the spring actually jumping out of contact with the end plates. Figure 10–5 is a photograph of a failure caused by such surging.

When helical springs are used in applications requiring a rapid reciprocating motion, the designer must be certain that the physical dimensions of the spring are not such as to create a natural vibratory frequency close to the frequency of the applied force; otherwise, resonance may occur, resulting in damaging stresses, since the internal damping of spring materials is quite low.

The governing equation for the translational vibration of a spring is the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{W}{kgl^2} \frac{\partial^2 u}{\partial t^2} \tag{10–24}$$

where k = spring rate

g = acceleration due to gravity

l = length of spring

W = weight of spring

x = coordinate along length of spring

u = motion of any particle at distance x

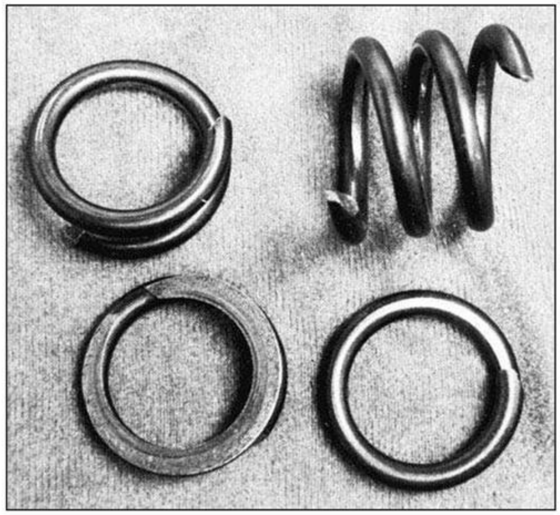

Figure 10–5

Valve-spring failure in an overrevved engine. Fracture is along the 45° line of maximum principal stress associated with pure torsional loading.

The solution to this equation is harmonic and depends on the given physical properties as well as the end conditions of the spring. The harmonic, *natural,* frequencies for a spring placed between two flat and parallel plates, in radians per second, are

$$\omega = m\pi\sqrt{\frac{kg}{W}} \qquad m = 1, 2, 3, \ldots$$

where the fundamental frequency is found for $m = 1$, the second harmonic for $m = 2$, and so on. We are usually interested in the frequency in cycles per second; since $\omega = 2\pi f$, we have, for the fundamental frequency in hertz,

$$f = \frac{1}{2}\sqrt{\frac{kg}{W}} \tag{10–25}$$

assuming the spring ends are always in contact with the plates.

Wolford and Smith[9] show that the frequency is

$$f = \frac{1}{4}\sqrt{\frac{kg}{W}} \tag{10–26}$$

where the spring has one end against a flat plate and the other end free. They also point out that Eq. (10–25) applies when one end is against a flat plate and the other end is driven with a sine-wave motion.

The weight of the active part of a helical spring is

$$W = AL\gamma = \frac{\pi d^2}{4}(\pi DN_a)(\gamma) = \frac{\pi^2 d^2 DN_a\gamma}{4} \tag{10–27}$$

where γ is the specific weight.

[9]J. C. Wolford and G. M. Smith, "Surge of Helical Springs," *Mech. Eng. News,* vol. 13, no. 1, February 1976, pp. 4–9.

The fundamental critical frequency should be greater than 15 to 20 times the frequency of the force or motion of the spring in order to avoid resonance with the harmonics. If the frequency is not high enough, the spring should be redesigned to increase k or decrease W.

10–9 Fatigue Loading of Helical Compression Springs

Springs are almost always subject to fatigue loading. In many instances the number of cycles of required life may be small, say, several thousand for a padlock spring or a toggle-switch spring. But the valve spring of an automotive engine must sustain millions of cycles of operation without failure; so it must be designed for infinite life.

To improve the fatigue strength of dynamically loaded springs, shot peening can be used. It can increase the torsional fatigue strength by 20 percent or more. Shot size is about $\frac{1}{64}$ in, so spring coil wire diameter and pitch must allow for complete coverage of the spring surface.

The best data on the torsional endurance limits of spring steels are those reported by Zimmerli.[10] He discovered the surprising fact that size, material, and tensile strength have no effect on the endurance limits (infinite life only) of spring steels in sizes under $\frac{3}{8}$ in (10 mm). We have already observed that endurance limits tend to level out at high tensile strengths (Fig. 6–17), p. 275, but the reason for this is not clear. Zimmerli suggests that it may be because the original surfaces are alike or because plastic flow during testing makes them the same. Unpeened springs were tested from a minimum torsional stress of 20 kpsi to a maximum of 90 kpsi and peened springs in the range 20 kpsi to 135 kpsi. The corresponding endurance strength components for infinite life were found to be

Unpeened:

$$S_{sa} = 35 \text{ kpsi (241 MPa)} \qquad S_{sm} = 55 \text{ kpsi (379 MPa)} \tag{10–28}$$

Peened:

$$S_{sa} = 57.5 \text{ kpsi (398 MPa)} \qquad S_{sm} = 77.5 \text{ kpsi (534 MPa)} \tag{10–29}$$

For example, given an unpeened spring with $S_{su} = 211.5$ kpsi, the Gerber ordinate intercept for shear, from Eq. (6–42), p. 298, is

$$S_{se} = \frac{S_{sa}}{1 - \left(\dfrac{S_{sm}}{S_{su}}\right)^2} = \frac{35}{1 - \left(\dfrac{55}{211.5}\right)^2} = 37.5 \text{ kpsi}$$

For the Goodman failure criterion, the intercept would be 47.3 kpsi. Each possible wire size would change these numbers, since S_{su} would change.

An extended study[11] of available literature regarding torsional fatigue found that for polished, notch-free, cylindrical specimens subjected to torsional shear stress, the maximum alternating stress that may be imposed without causing failure is *constant* and independent of the mean stress in the cycle provided that the maximum stress range does not equal or exceed the torsional yield strength of the metal. With notches and abrupt section changes this consistency is not found. Springs are free of notches and surfaces are often very smooth. This failure criterion is known as the *Sines failure criterion* in torsional fatigue.

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[10]F. P. Zimmerli, "Human Failures in Spring Applications," *The Mainspring*, no. 17, Associated Spring Corporation, Bristol, Conn., August–September 1957.

[11]Oscar J. Horger (ed.), *Metals Engineering: Design Handbook*, McGraw-Hill, New York, 1953, p. 84.

In constructing certain failure criteria on the designers' torsional fatigue diagram, the torsional modulus of rupture S_{su} is needed. We shall continue to employ Eq. (6–54), p. 309, which is

$$S_{su} = 0.67 S_{ut} \tag{10–30}$$

In the case of shafts and many other machine members, fatigue loading in the form of completely reversed stresses is quite ordinary. Helical springs, on the other hand, are never used as both compression and extension springs. In fact, they are usually assembled with a preload so that the working load is additional. Thus the stress-time diagram of Fig. 6–23*d*, p. 293, expresses the usual condition for helical springs. The worst condition, then, would occur when there is no preload, that is, when $\tau_{\min} = 0$.

Now, we define

$$F_a = \frac{F_{\max} - F_{\min}}{2} \tag{10–31a}$$

$$F_m = \frac{F_{\max} + F_{\min}}{2} \tag{10–31b}$$

where the subscripts have the same meaning as those of Fig. 7–23*d* when applied to the axial spring force F. Then the shear stress amplitude is

$$\tau_a = K_B \frac{8 F_a D}{\pi d^3} \tag{10–32}$$

where K_B is the Bergsträsser factor, obtained from Eq. (10–6), and corrects for both direct shear and the curvature effect. As noted in Sec. 10–2, the Wahl factor K_W can be used instead, if desired.

The midrange shear stress is given by the equation

$$\tau_m = K_B \frac{8 F_m D}{\pi d^3} \tag{10–33}$$

EXAMPLE 10–4

An as-wound helical compression spring, made of music wire, has a wire size of 0.092 in, an outside coil diameter of $\frac{9}{16}$ in, a free length of $4\frac{3}{8}$ in, 21 active coils, and both ends squared and ground. The spring is unpeened. This spring is to be assembled with a preload of 5 lbf and will operate with a maximum load of 35 lbf during use.

(*a*) Estimate the factor of safety guarding against fatigue failure using a torsional Gerber fatigue failure criterion with Zimmerli data.

(*b*) Repeat part (*a*) using the Sines torsional fatigue criterion (steady stress component has no effect), with Zimmerli data.

(*c*) Repeat using a torsional Goodman failure criterion with Zimmerli data.

(*d*) Estimate the critical frequency of the spring.

Solution

The mean coil diameter is $D = 0.5625 - 0.092 = 0.4705$ in. The spring index is $C = D/d = 0.4705/0.092 = 5.11$. Then

$$K_B = \frac{4C + 2}{4C - 3} = \frac{4(5.11) + 2}{4(5.11) - 3} = 1.287$$