Mechanics of Materials

Chapter 5

Stresses In Beams

5.1 Introduction

- In previous chapters, the stresses in bars caused by axial loading and torsion. Here consider the third fundamental loading : bending.
- Make certain simplifying assumptions. the resulting equations have served well in the design of straight, elastic beams

5.2 *Bending Stress*

a. *Simplifying assumptions*

- The stresses caused by the bending moment are known as ***bending stress***, or ***flexure stresses***. The relationship between these stresses and the bending moment is called the ***flexure formula***.

- In deriving the flexure formula, make the following assumptions:

  - The beam has an axial plane of symmetry, which we take to be the *xy*-plane (see Fig. 5.1).

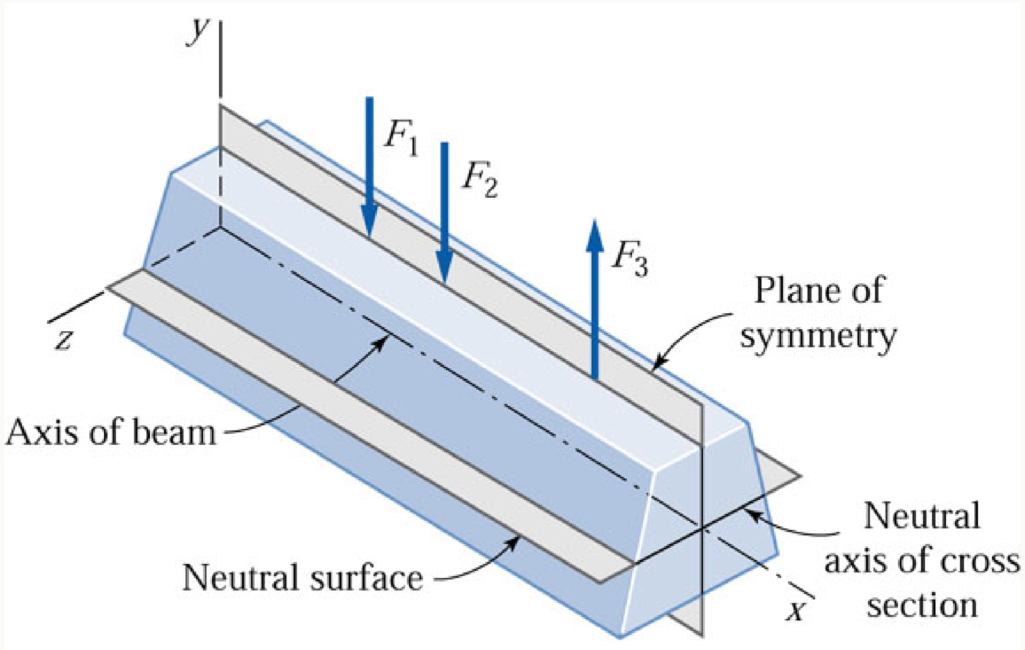


Figure 5.1 Symmetrical beam with loads lying in the plane of symmetry.

- The applied loads (such as F_1,F_2 and F_3 in Fig.5.1) lie in the plane of the symmetry and are perpendicular to the axis of the beam (the *x*-axis).The axis of the beam **bends but does not stretch** (the axis lies some where in the plane of symmetry; its location will be determined later).

- Plane sections of the beam remain plane (do not warp) and perpendicular to the deformed axis of the beam. Change in the cross-sectional dimensions of the beam are negligible.

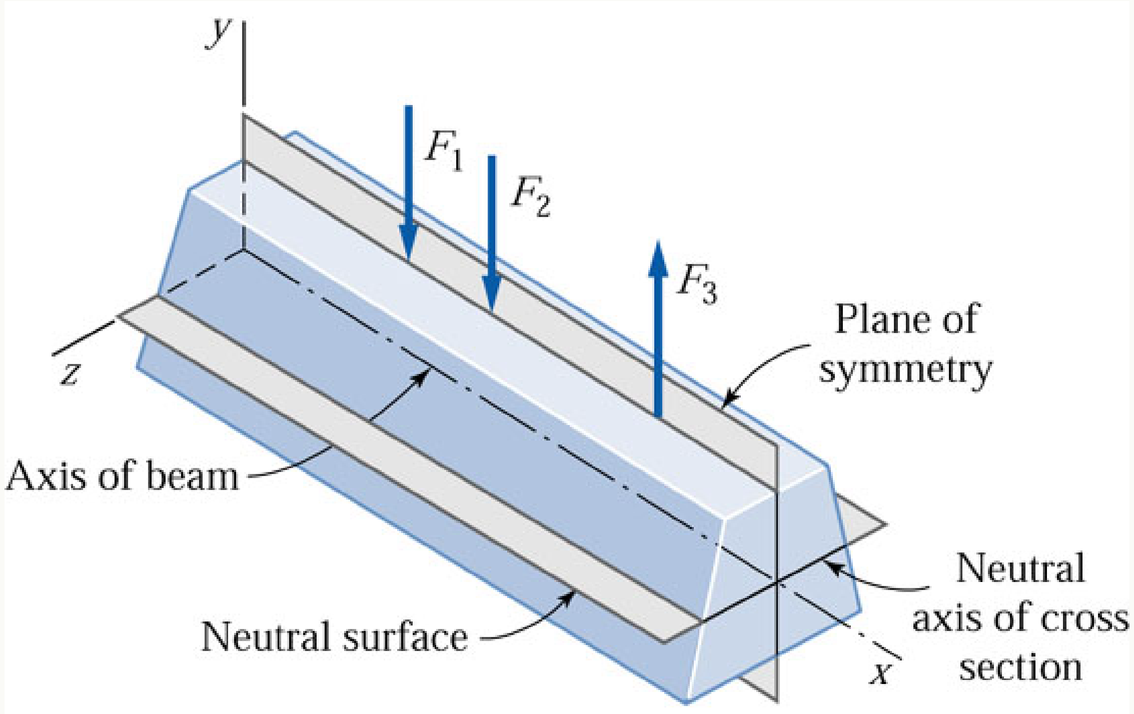
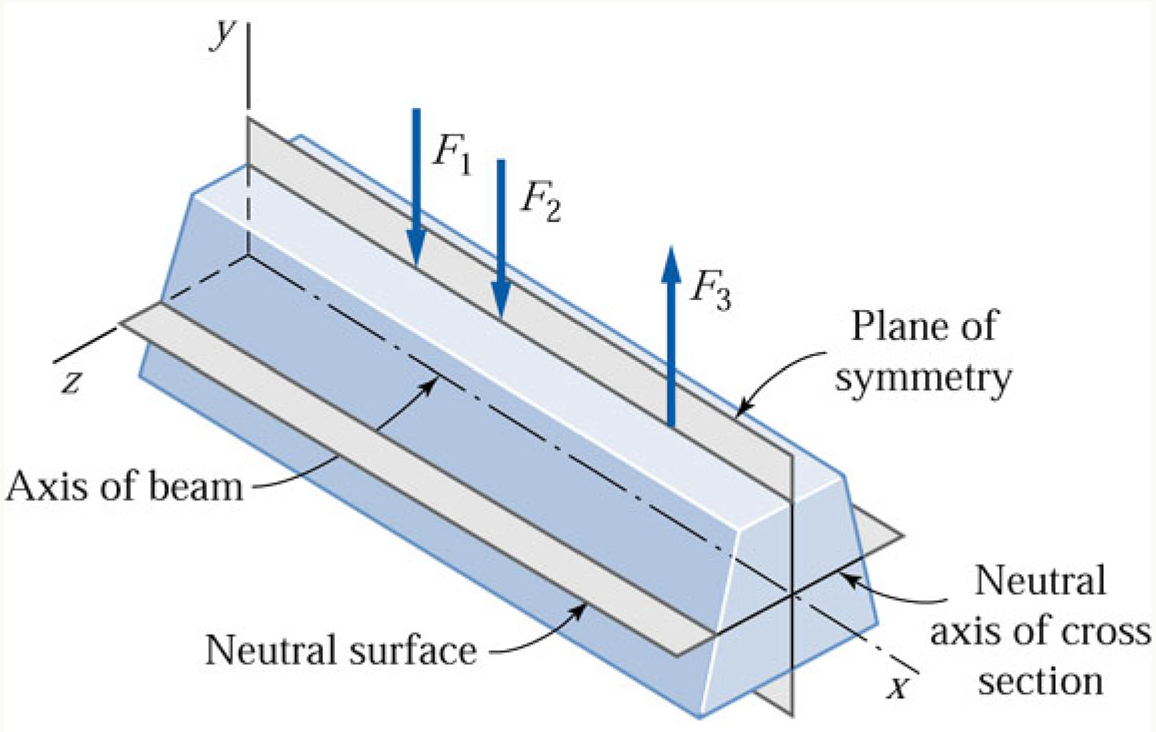
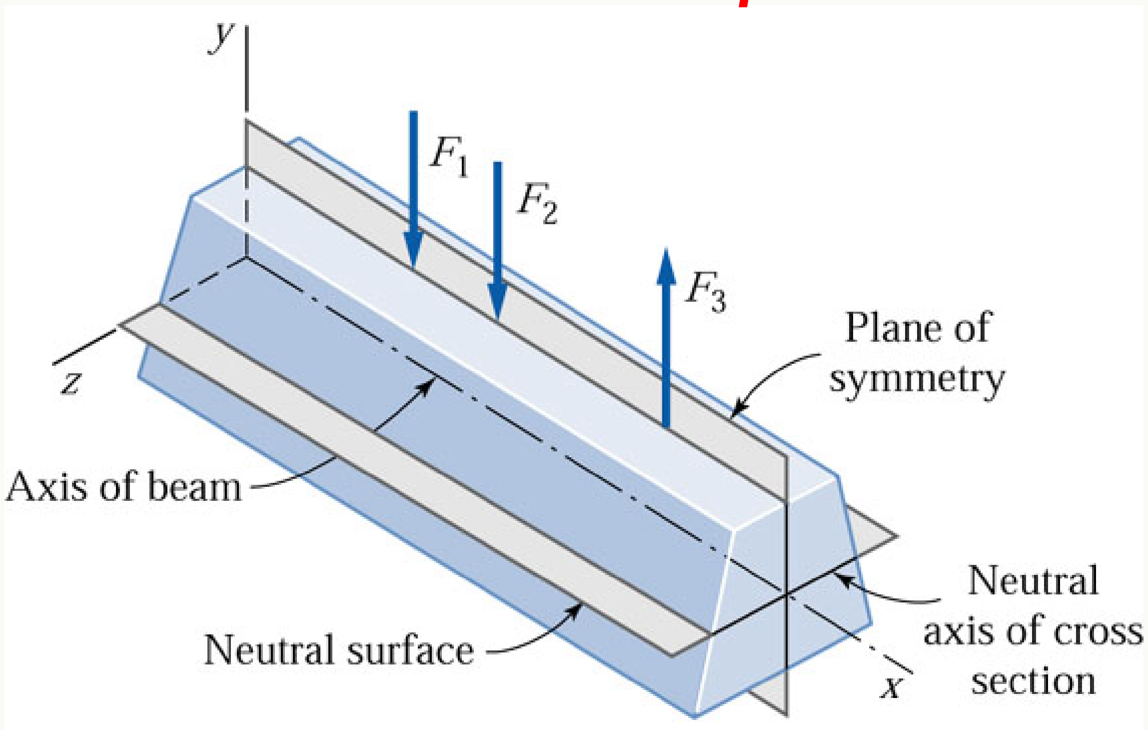


Figure 5.1 Symmetrical beam

- Because the shear stresses caused by the vertical shear force will distort (warp) an originally plane section, we are limiting our discussion here to the deformations caused by the bending moment alone.
- the deformations due to the vertical shear force are negligible in the slender beams compared to the deformations caused by bending .



- The above assumptions lead us to the following conclusion: *Each cross section of the beam rotates as a rigid entity about a line called the neutral axis of the cross section.*
- The ***neutral axis*** passes through the axis of the beam and is perpendicular to **the plane of symmetry**, as shown in Fig. 5.1. The *xz*-plane that contains the neutral axes of all the cross sections is known as **the *neutral surface* of the beam**.



b. *Compatibility*

- The neutral surface becomes curved upon deformation, as indicated in Fig.5.2.
- The longitudinal fibers lying on the neutral surface are undeformed, whereas the fibers above the surface are compressed and the fibers below are stretched.

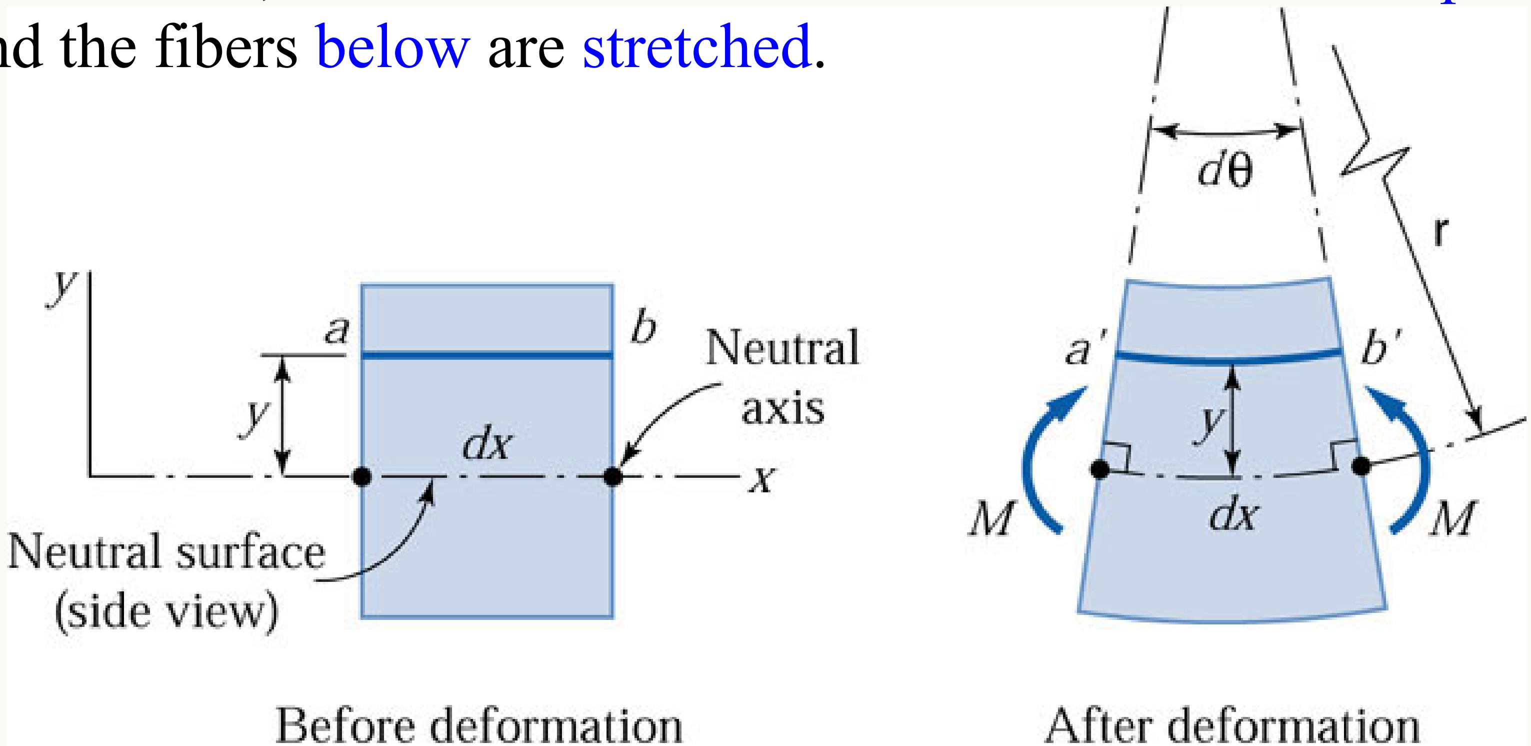
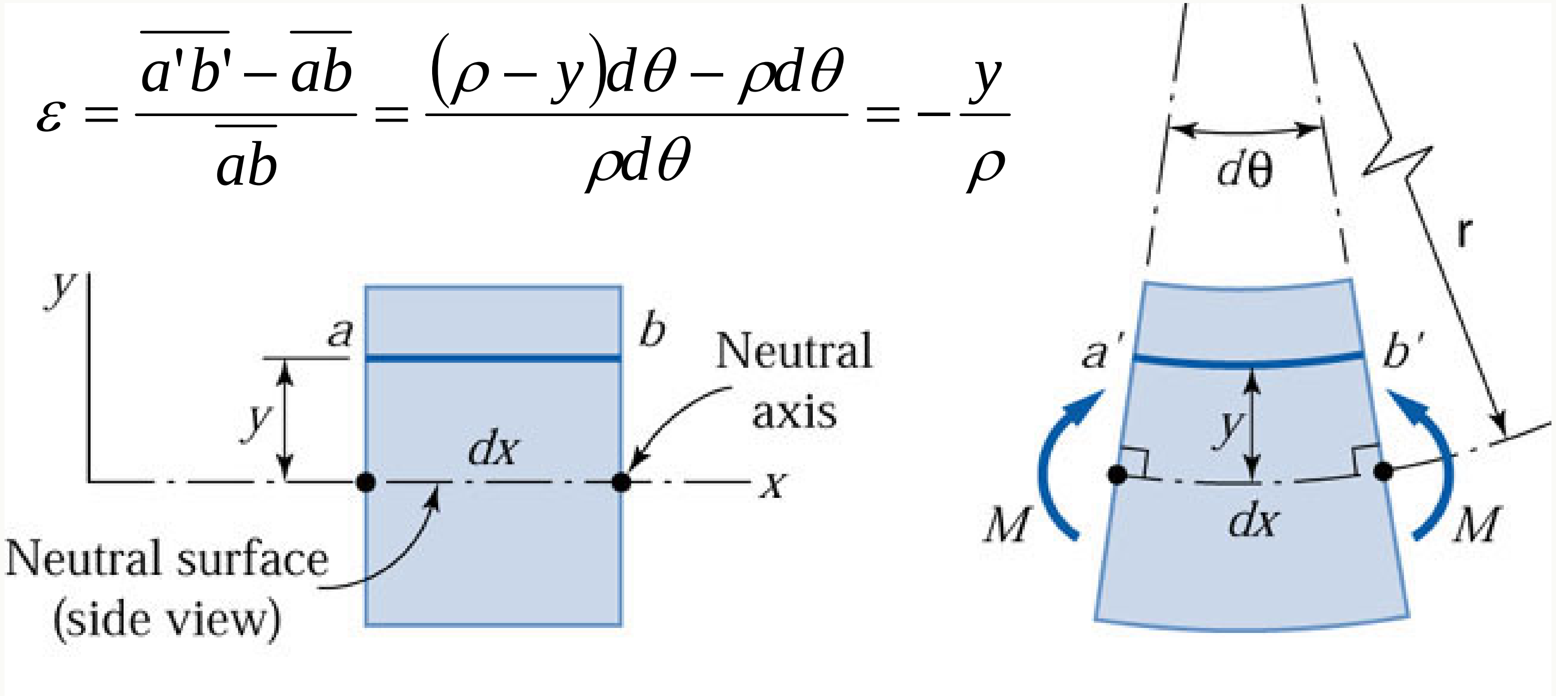


Figure 5.2 Deformation of an infinitesimal beam segment.

- The fiber form are arc $a'\,b'$ of radius (ρ-y) , subtended by the angle d θ , its deformed length is $\overline{a'b'} = (\rho - y)d\theta$
- The original length of this fiber is $\overline{ab} = dx = \rho d\theta$. The normal strain ε of the fiber

$$\varepsilon = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho}$$



Before deformation

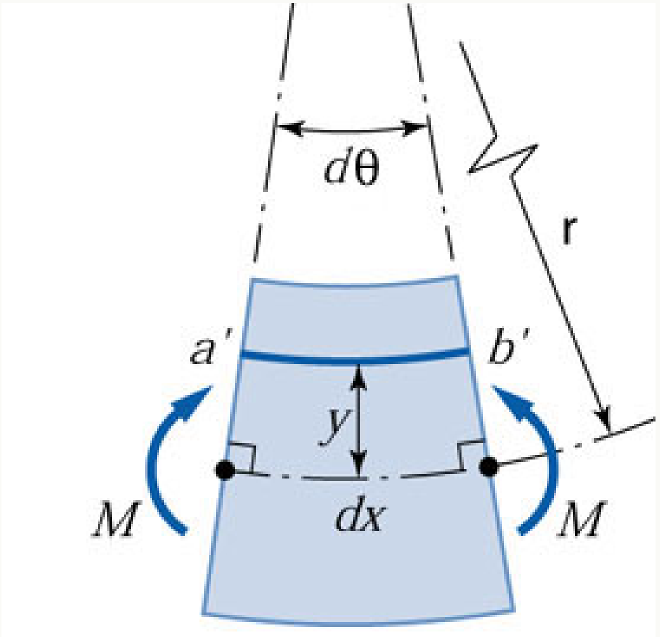
After deformation

- Assuming that the stress is less than the proportional limit of the material we can obtain the normal stress in the fiber ab from Hook′ s law:

$$\boxed{\sigma = E\varepsilon = -\frac{E}{\rho}y} \tag{5.1}$$

Equation (5.1) shown that the normal stress of a longitudinal fiber is proportional to the distance y of the fiber from the neutral surface.

- The negative sign indicates that positive bending moment causes compressive stress when *y* is positive (fiber above the neutral surface) and tensile stress when *y* is negative (fiber below the neutral surface).



c. Equilibrium

- Figure 5.3 shows the normal force acting on the infinitesimal area dA of the cross section is $dP = \sigma\, dA$. Substituting $\sigma = -(E/\rho)y$,

$$dP = -\frac{E}{\rho} y dA \qquad \text{(a)}$$

  Where y is the distance of dA from the neutral axis (NA).

- The resultant of the normal stress distribution over the cross section must be equal to the bending moment M acting about the neutral axis (z-axis).

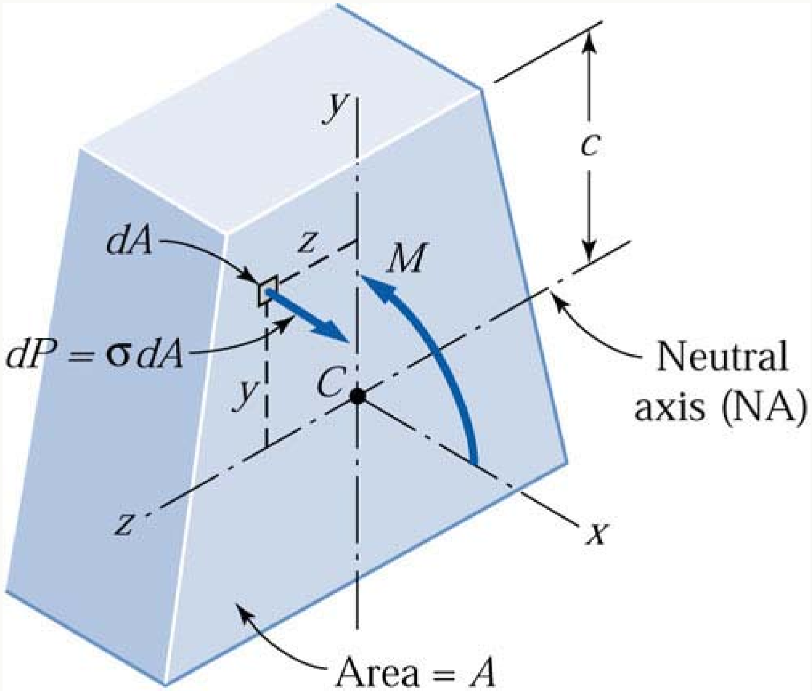


Figure 5.3 Calculating the resultant of the Normal stress acting on the cross section. Resultant is a couple Equal to the internal bending moment of *M*.

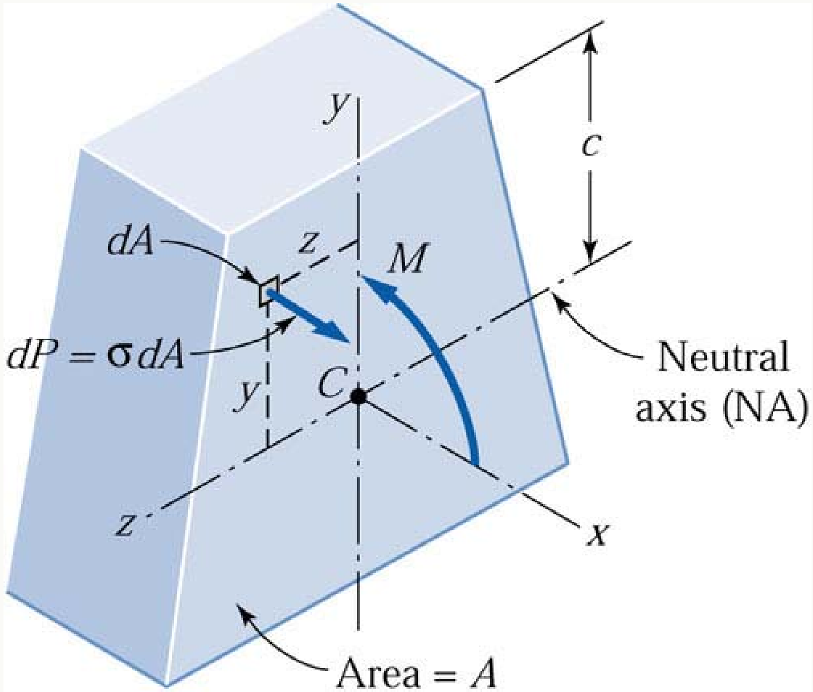
- In other work, $\boxed{-\int_A y\,dp = M}$

  where the integral is taken over the entire cross-sectional area A

- the resultant axial force and the resultant bending moment about the y-axis must be zero; that is,

  $\boxed{\int_A dP = 0}$ and $\boxed{\int_A z\,dP = 0}$

  These three equilibrium equations are developed in detail below.



Resultant Axial Force Must Vanish The condition for zero axial force is

$$\int_A dp = -\frac{E}{\rho}\int_A y\,dA = 0$$

- Because E / ρ ≠ 0, this equation can be satisfied only if

$$\boxed{\int_A y\,dA = 0} \qquad \text{(b)}$$

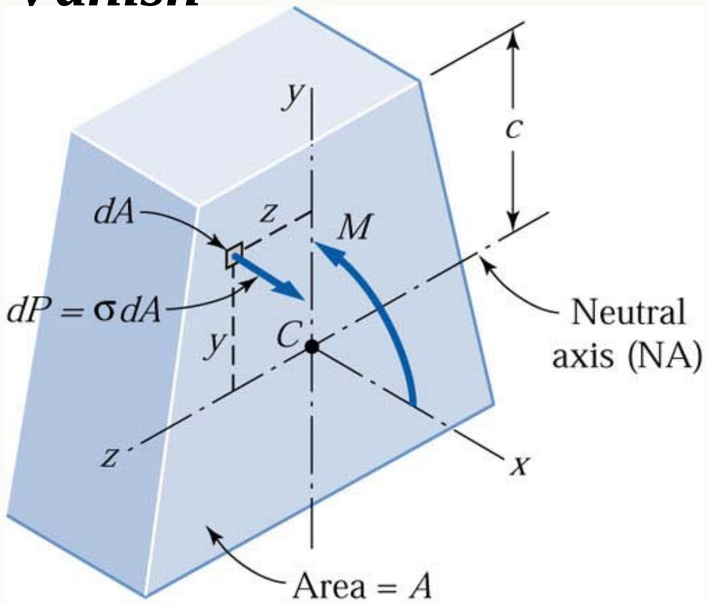
The integral in Eq.(b) is the first moment of the cross-sectional area about the neutral axis. It can be zero only if the neutral axis passes through **centroid C of the cross-sectional area**.

Resultant Moment About y-Axis Must Vanish

This condition is

$$\boxed{\int_A z\,dP = -\frac{E}{\rho}\int_A zy\,dA = 0} \qquad \text{(c)}$$

The integral in Eq.(b) is **the product of inertia of the cross-sectional area**.



Resultant Moment About the Neutral Axis Must Equal M

Equating the resultant moment about the *z*-axis to *M*

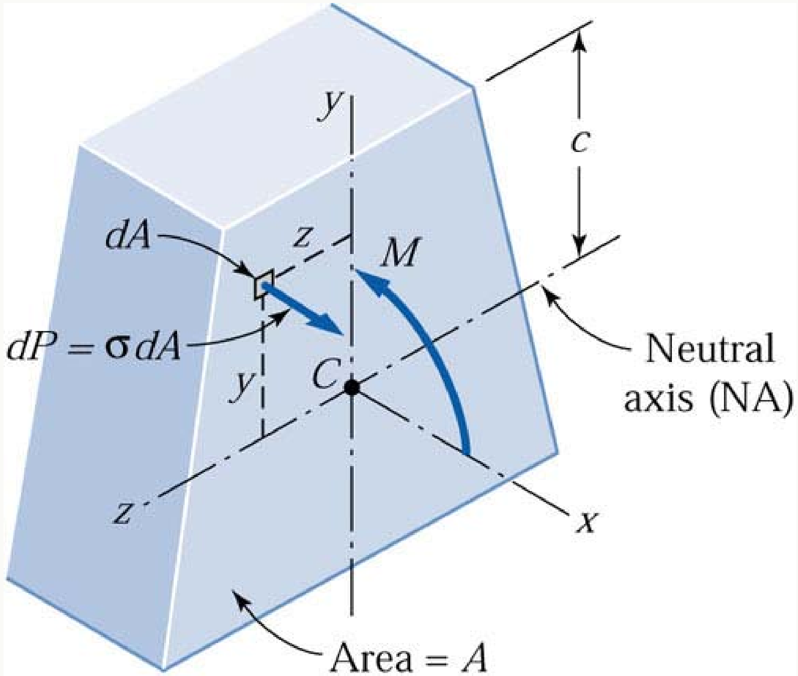
$$\boxed{-\int_A y\,dp = M} \qquad \boxed{-\int_A y\,dp = \frac{E}{\rho}\int_A y^2 dA = M}$$

Recognizing that $\boxed{\int_A y^2 dA = I}$ is the moment of inertia of the cross-sectional area about the neutral axis (the z-axis), we obtain the ***moment curvature relationship***

$$\boxed{M = \frac{EI}{\rho}} \qquad (5.2a)$$

A convenient form of this equation is

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \qquad (5.2b)$$



d. ***Flexure formula; section modulus***

- Substituting the expression for $1/\rho$ from Eq.(5.2) into Eq. (5.1), we get the *flexure formula* :

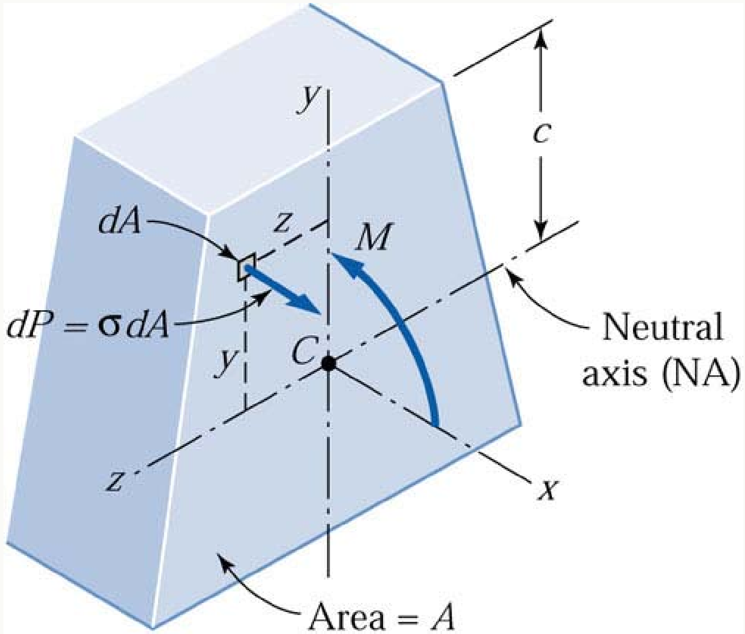
$$\boxed{\sigma = -\frac{My}{I}} \qquad (5.3)$$

Note that a positive bending moment M causes negative (compressive) stress above the neutral axis and positive (tensile) stress below the neutral axis

- The **maximum value** of bending stress without regard to its sign is given by

$$\boxed{\sigma_{\max} = \frac{[M_{\max}]c}{I}} \qquad (5.4a)$$

where c is the distance from the neutral axis to the outermost point of the cross section.



Equation (5.4a) is frequently written in the form

$$\boxed{\sigma_{max} = \frac{[M_{max}]}{S}} \qquad (5.4b)$$

where **$S = I / c$** is called ***the section modulus* of the beam**. The dimension of S is $[L^3]$, so that its units are in.^{3}, mm^3, and so on. The formulas for the section moduli of common cross sections are given in Fig. 5.4.

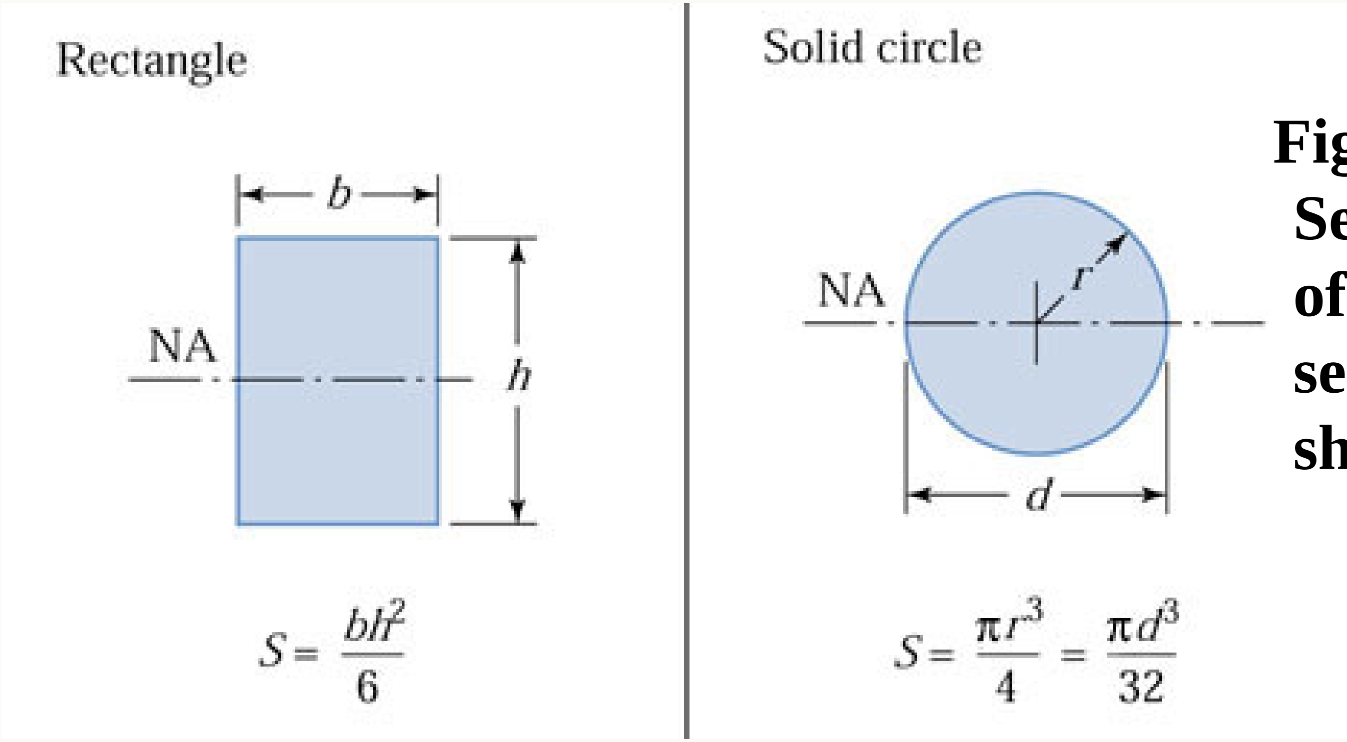
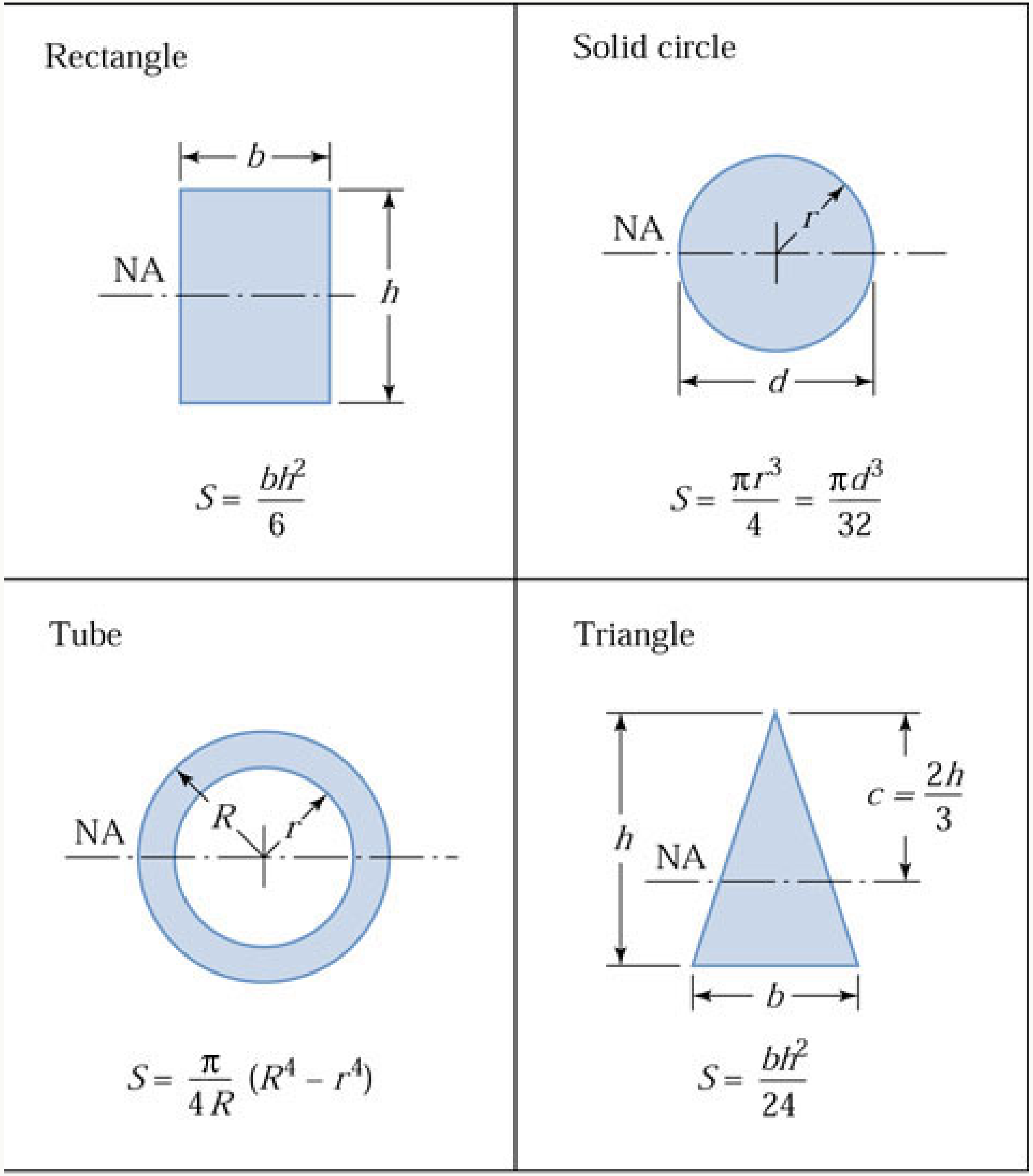


Figure 5.4 Section moduli of simple cross sectional shapes.



- The section moduli of standard structural shapes are listed in various handbooks; an abbreviated list is given in Appendix B.

Figure 5.4 Section moduli of simple cross sectional shapes.

e. Procedures for determining bending stresses

Stress at a Given Point

- Use the method of sections to determine the bending moment M at the cross section containing the given point.
- Determine the location of the neutral axis.
- Compute the moment of inertia I of the cross- sectional area about the neutral axis. (If the beam is standard structural shape, its cross- sectional properties are listed in Appendix B. P501)
- Determine the y-coordinate of the given point. Note that y is positive if the point lies above the neutral axis and negative if it lies below the neutral axis.
- Compute the bending stress from σ = -My / I. If correct sign are used for *M* and *y*, the stress will also have the correct sign (tension positive compression negative).

Maximum Bending Stress: Symmetric Cross Section

If the neutral axis is an axis of symmetric of the cross section, the maximum tensile and compression bending stresses are equal in magnitude and occur at the section of the largest bending moment. The following procedure is recommended for determining the maximum bending stress in a prismatic beam:

- Draw the bending moment diagram by one of the methods described in Chapter 4. Identify the bending moment M_{max} that has the largest magnitude (disregard the sign)
- Compute the moment of inertia I of the cross- sectional area about the neutral axis. (If the beam is a standard structural shape, its cross- sectional properties are listed in Appendix B.)
- Calculate the maximum bending stress from $\sigma_{max} = [M_{max}]c / I$, where c is the distance from the neutral axis to the top or bottom of the cross section .

Maximum Tensile and Compressive Bending Stresses:

Unsymmetrical Cross Section

If the neutral axis is not an axis of symmetry of the cross section, the maximum tensile and compressive bending stresses may occur at different sections.

- Draw the bending moment diagram. Identify the largest positive and negative bending moments.
- Determine the location of the neutral axis and record the distances c_{top} and c_{bot} from the neutral axis to the top and bottom of the cross section.
- Compute the moment of inertia I of the cross section about the neutral axis.

- Calculate the bending stresses at the top and bottom of the cross section where the largest positive bending moment occurs from $\sigma = -My / I$.

✓ At the top of the cross section, where $y = c_{top}$, we obtain $\sigma_{top} = -Mc_{top} / I$.

✓ At the bottom of the cross section, we have $y = -c_{bot}$, so that $\sigma_{bot} = Mc_{bop} / I$.

- Repeat the calculations for the cross section that carries the largest negative bending moment.
- Inspect the four stresses thus computed to determine the largest tensile (positive) and compressive (negative) bending stresses in the beam.

Note on Units

the units of terms in the flexure formula $\sigma = -My / I$.

- In the U.S. Customary system, M is often measured in pound-feet and the cross sectional properties in inches, It is recommended that you convert M into lb·in. and compute σ in lb/in.^{2} (psi). Thus, the units in the flexure formula become

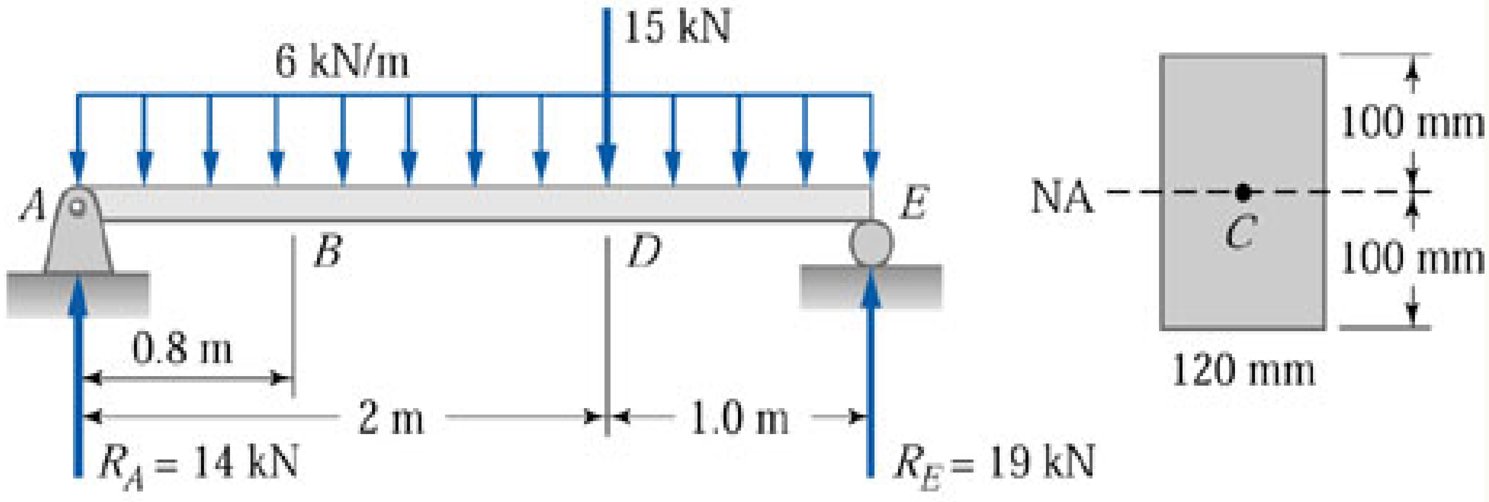
$$\sigma\left[lb / in.^2\right] = \frac{M\left[lb \cdot in.\right] y\left[in.\right]}{I\left[in.^4\right]}$$

- In SI system, M is usually expressed in N · m, whereas the cross-sectional dimensions are in mm. To obtain σ in N/m^2 (Pa), he cross sectional properties must be converted to meters, so that the units in the flexure equation are

$$\sigma\left[N / m^2\right] = \frac{M\left[N \cdot m\right] y\left[m\right]}{I\left[m^4\right]}$$

Sample Problem 5.1

The simply supported beam in Fig. (a) has a rectangular cross section 120 mm wide and 200 mm high. (1) Compute the maximum bending stress in the beam. (2) Sketch the bending stress distribution over the cross section on which the maximum bending stress occurs. (3) Compute the bending stress at a point on section *B* that is 25 mm below the top of the beam.



(a)

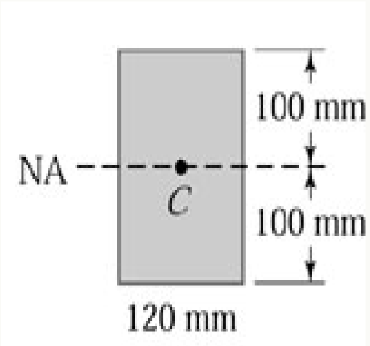
Solution

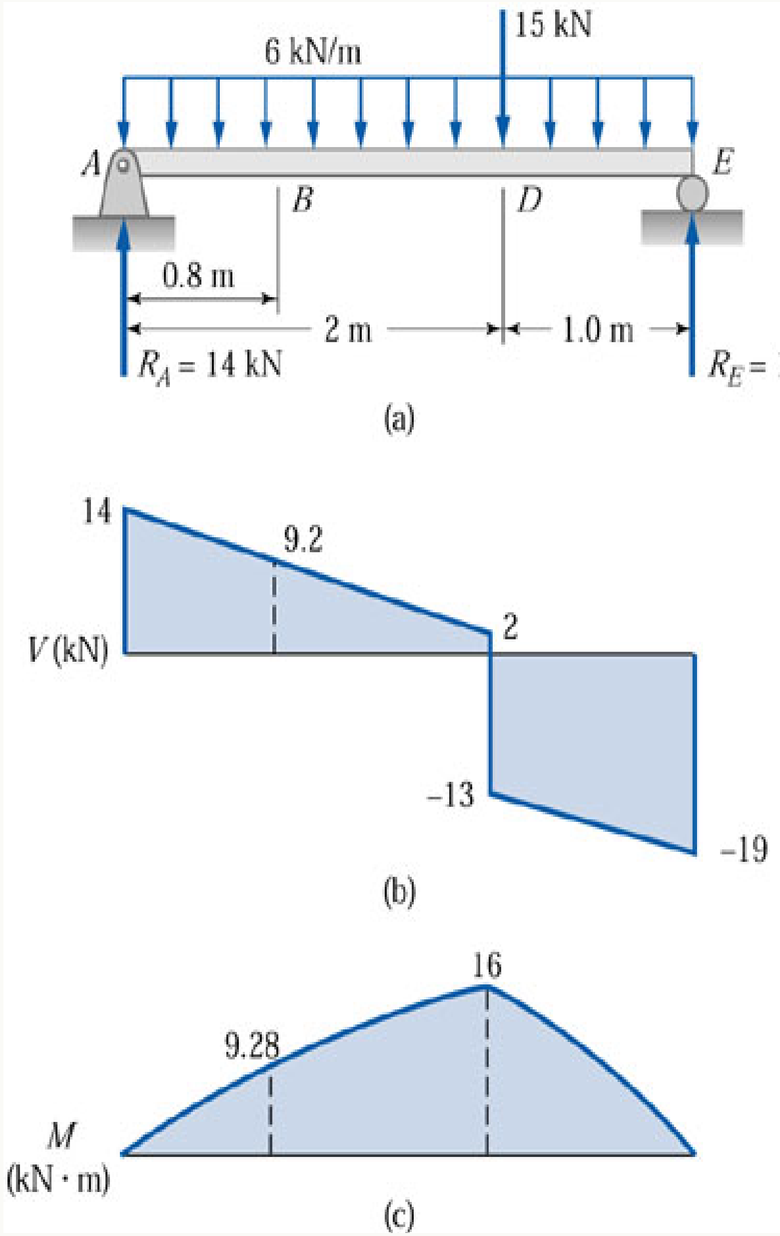
Preliminary Calculations

The shear force and bending moment diagrams. $M_{\max} = +16$ kN·m, occurring at *D*. The neutral axis (NA) is an axis of symmetry of the cross section as shown in Fig. (a). The moment of inertia of the cross section about the neutral axis is

$$I = \frac{bh^3}{12} = \frac{0.12(0.2)^3}{12} = 800 \times 10^{-6}\, m^4$$

and the distance c between the neutral axis and the top (or bottom) of the cross section is c = 100 mm = 0.1 m.





(a)

(b)

(c)

Part 1

The maximum bending stress in the beam on the cross section that carries the largest bending moment, which is the section at *D*.

$$\sigma_{max} = \frac{[M_{max}]c}{I} = \frac{(16\times 10^{3})(0.1)}{80.0\times 10^{-6}} = 20.0\times 10^{6}\,Pa = 20.0MPa \quad \text{Answer}$$

Part 2

The stress distribution on the cross section at *D* is shown in Fig. (d)

(i) The bending stress varies linearly with distance from the neutral axis;

(ii) Because M_{max} is positive, the top half of the cross section is in compression and the bottom half is in tension.

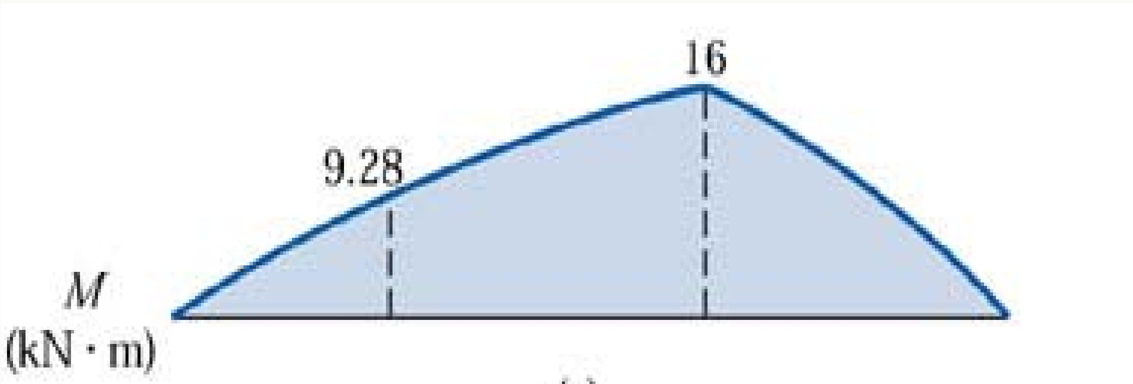
(iii)Due to symmetry of the cross section about the neutral axis, the maximum tensile and compressive stresses are equal in magnitude.

Part 3

From Fig. (c) we see that the bending moment at section *B* is $M = +9.28\ \text{kN}\cdot\text{m}$. The *y*-coordinate of the point that lies 25 mm below the top of the beam is $y = 100 - 25 = 75$ mm $= 0.075$ m.

$$\sigma = -\frac{My}{I} = -\frac{(9.28\times10^{3})(0.075)}{80.0\times10^{-6}} = -8.70\times10^{6}\,Pa = -8.70\,MPa \qquad \text{Answer}$$

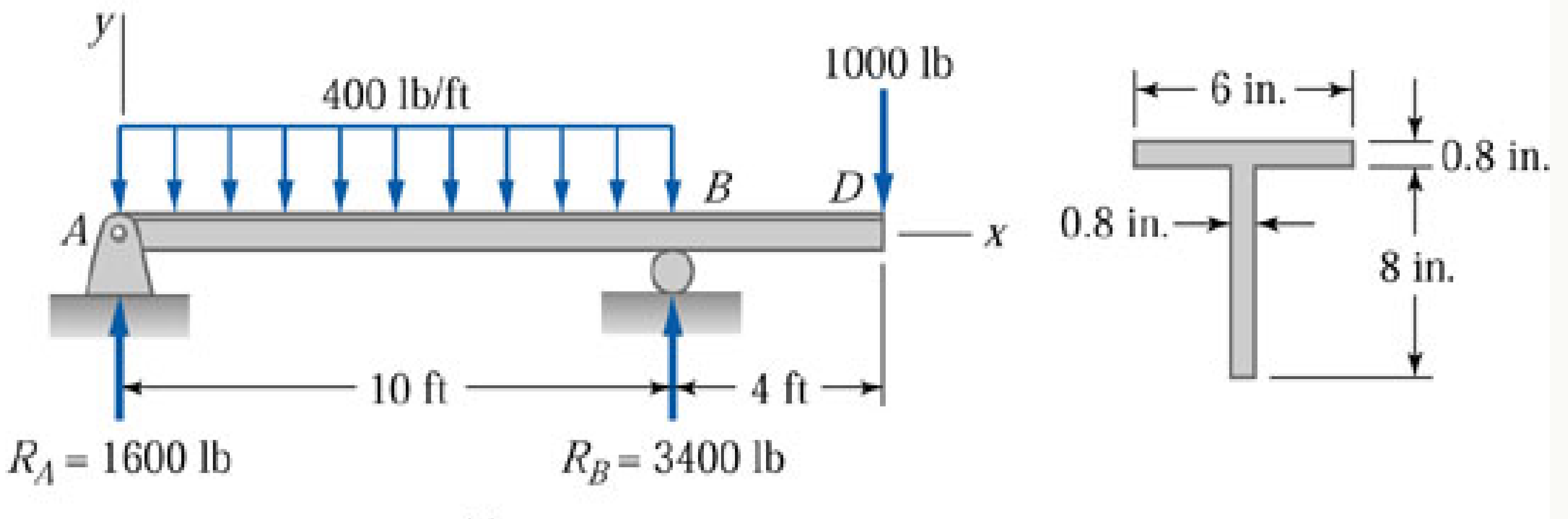
The negative sign indicates that this bending stress is compressive, which is expected because the bending moment is positive and the point of interest lie above the neutral axis.



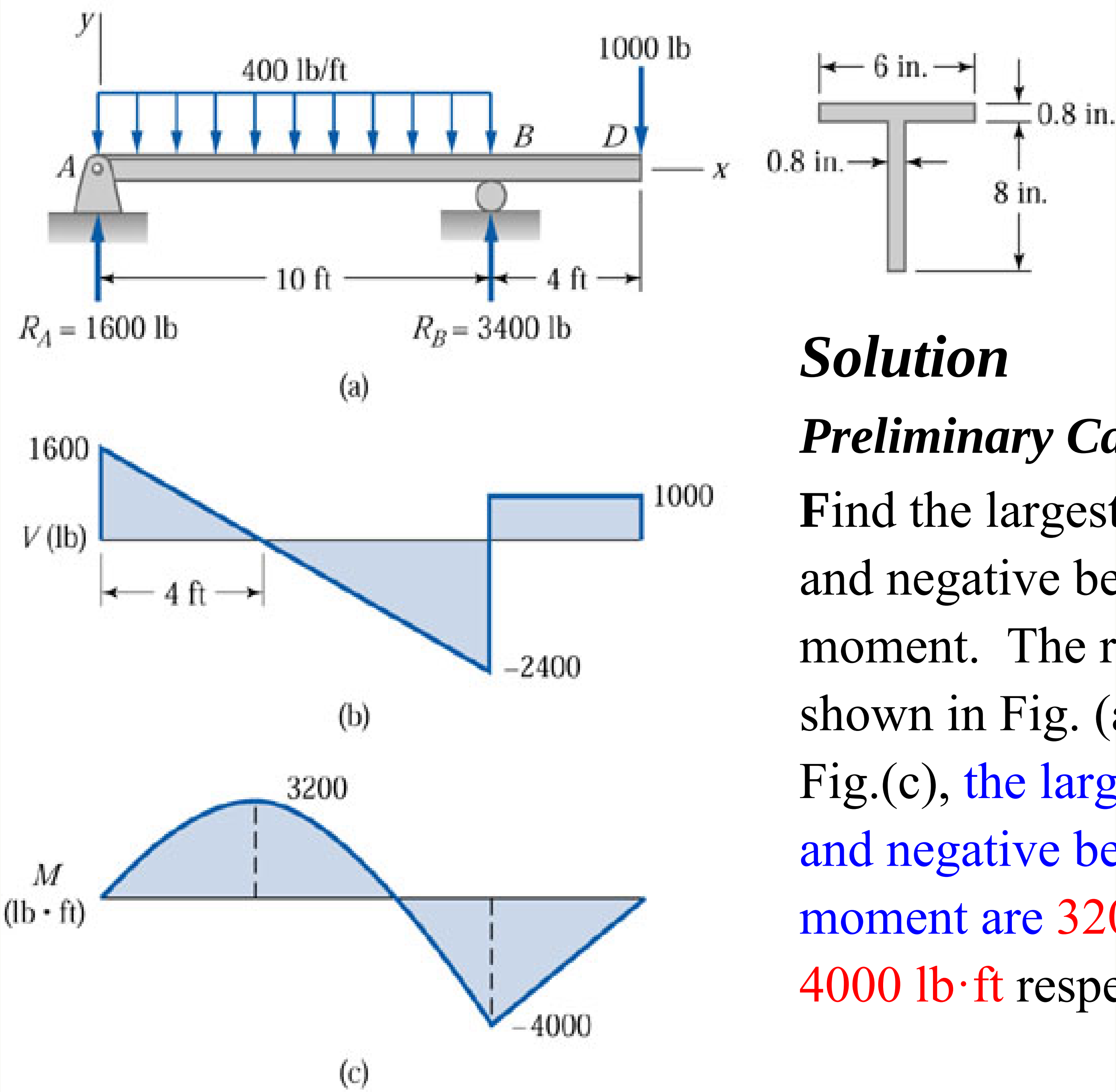
(c)

Sample Problem **5.2**

The simply supported beam in Fig. (a) has the T-shaped cross section shown. Determine the values and locations of the maximum tensile and compressive bending stresses.



(a)



Solution

Preliminary Calculations

Find the largest positive and negative bending moment. The results are shown in Fig. (a)-(c). From Fig.(c), the largest positive and negative bending moment are 3200 lb·ft and 4000 lb·ft respectively.

As shown in Fig.(d), the cross section to be composed of the two rectangles with areas $A_1 = 0.8(8) = 6.4$ in.^{2} and $A_2 = 0.8\ (6) = 4.8$ in.^{2}. The centroidal coordinates of the areas are $\overline{y_1} = 4in.$ and $\overline{y_2} = 8.4in.$, , measured from the bottom of the cross section. The coordinate $\bar{y}$ of the centroid C of the cross section is

$$\bar{y} = \frac{A_1\overline{y_1} + A_2\overline{y_2}}{A_1 + A_2} = \frac{6.4(4) + 4.8(8.4)}{6.4 + 4.8} = 5.886in.$$

Compute the moment of inertia I of the cross-sectional area about the neutral axis. Using **the parallel-axis theorem**, $I = \sum\left[\overline{I_i} + A_i\left(\overline{y_i} - \bar{y}\right)^2\right]$, where $\overline{I_i} = b_i h_i^3/12$ is the moment of inertia of a rectangle about its own centroidal axis Thus,

$$I = \left[\frac{0.8(8)^3}{12} + 6.4(4 - 5.886)^2\right] + \left[\frac{6(0.8)^3}{12} + 4.8(8.4 - 5.886)^2\right] = 87.49in.^4$$

Maximum Bending stresses

The distances from the neutral axis to the top and the bottom of the cross section are $c_{top} = 8.8 - \overline{y} = 8.8 - 5886 = 2.914 in.$ and $c_{bot} = \overline{y} = 5.886 in.$, as shown in Fig.(c). Because these distances are different, we must investigate stresses at two locations: at x = 4 ft (where the largest positive bending moment occurs) and at x = 10 ft (where the largest negative bending moment occurs).

Stresses at x = 4 ft The bending moment at this section is M = +3200 lb · ft causing compression above the neutral axis and tension below the axis. The resulting bending stresses at the top and bottom of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = -\frac{(3200 \times 12)(2.914)}{87.49} = -1279\,psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = \frac{(3200 \times 12)(-5.886)}{87.49} = 2580\,psi$$

Stresses at x = 10 ft The bending moment at this section is M = -4000lb · ft, resulting in tension the neutral axis and compression below the neutral axis. The corresponding bending stresses at the extremities of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = \frac{(-4000 \times 12)(2.914)}{87.49} = 1599\, psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = \frac{(-4000 \times 12) - (5.886)}{87.49} = -3230\, psi$$

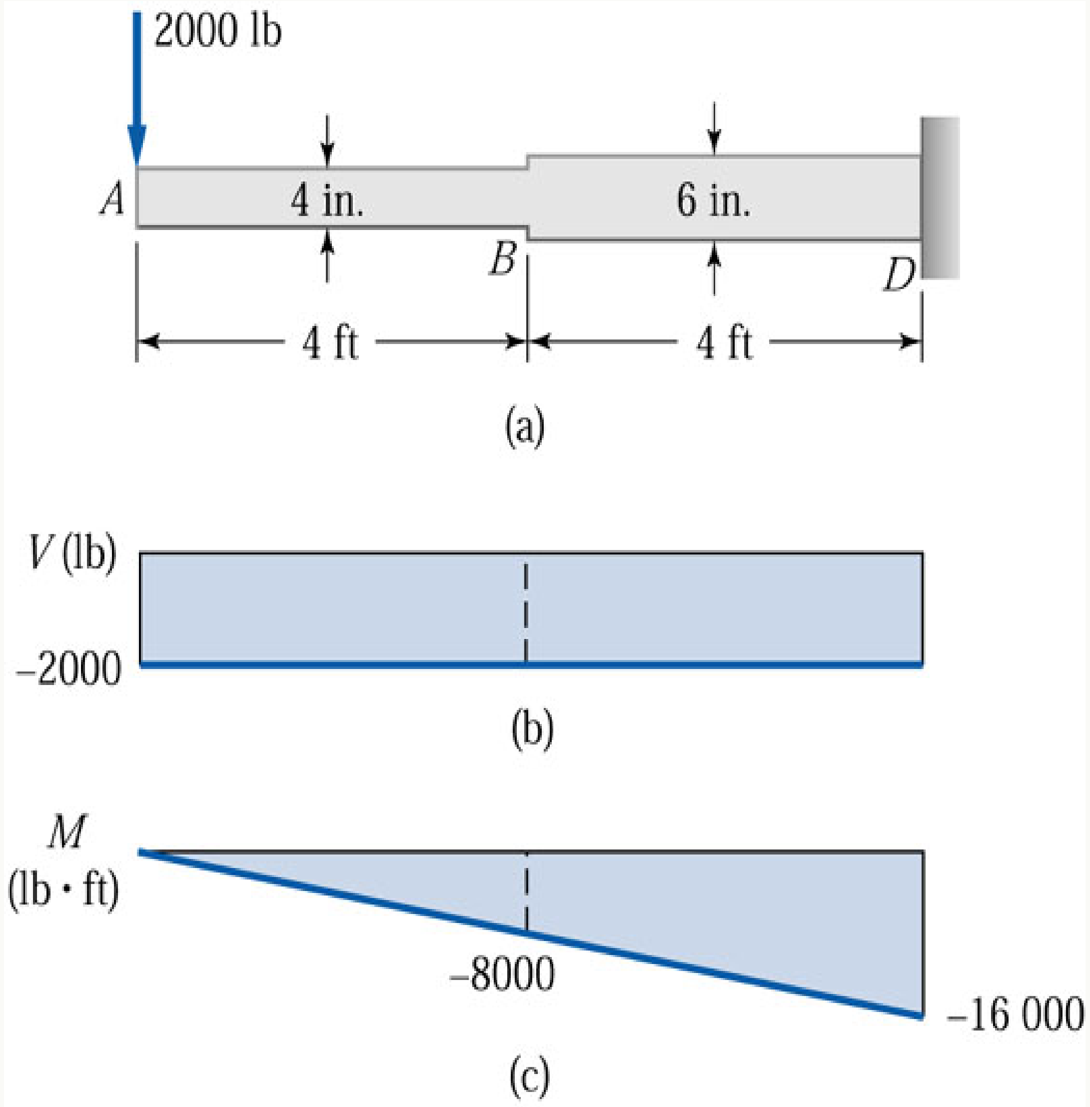
Inspecting the above results, we conclude that the maximum tensile and compressive stresses in the beam are

$(\sigma_T)_{max} =$ 2580 psi (bottom of the section at x = 4 ft)

$(\sigma_c)_{max} =$ 3230 psi (bottom of the section at x = 10 ft)

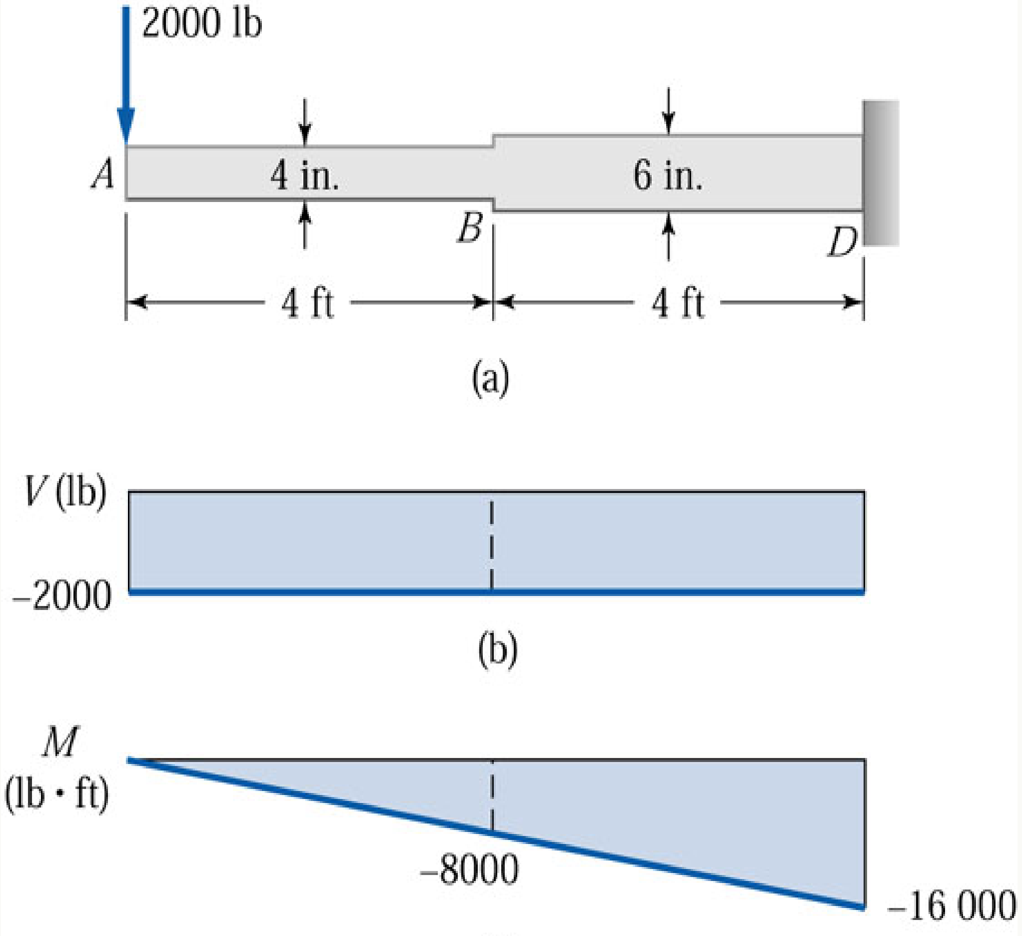
Sample Problem **5.3**

The cantilever beam in Fig. (a) is composed of two segments with rectangular cross sections. The width of the each section is 2 in., but the depths are different, as shown in the figure. Determine the maximum bending stress in the beam.



Solution

Because the cross section of the beam is not constant, the maximum stress occurs either at the section just to the left of B (M_B = - 8000 lb · ft) or at the section at D (M_D = - 16000 lb · ft). the section moduli of the two segments are



(a)
(b)
(c)

$$S_{AB} = \frac{bh^2{}_{AB}}{6} = \frac{(2)(4)^2}{6} 5.333 in.^3$$

$$S_{BD} = \frac{bh^2{}_{BD}}{6} = \frac{(2)(6)^2}{6} 12.0 in.^3$$

From Eq. (5.4b) the maximum bending stresses on the two cross sections of the interest are

$$(\sigma_B)_{\max} = \frac{[M_B]}{S_{AB}} = \frac{8000 \times 12}{5.333} = 18000\, psi$$

$$(\sigma_D)_{\max} = \frac{[M_D]}{S_{BD}} = \frac{16000 \times 12}{12.0} = 16000\, psi$$

Comparing the above values, we find that the maximum bending stress in the beam is

$\sigma_{\max}$ = 18000 psi (on the cross section just to the left of *B*)

Answer

❖ This is an example where the maximum bending stress occurs on a cross section at the bending moment is not maximum.

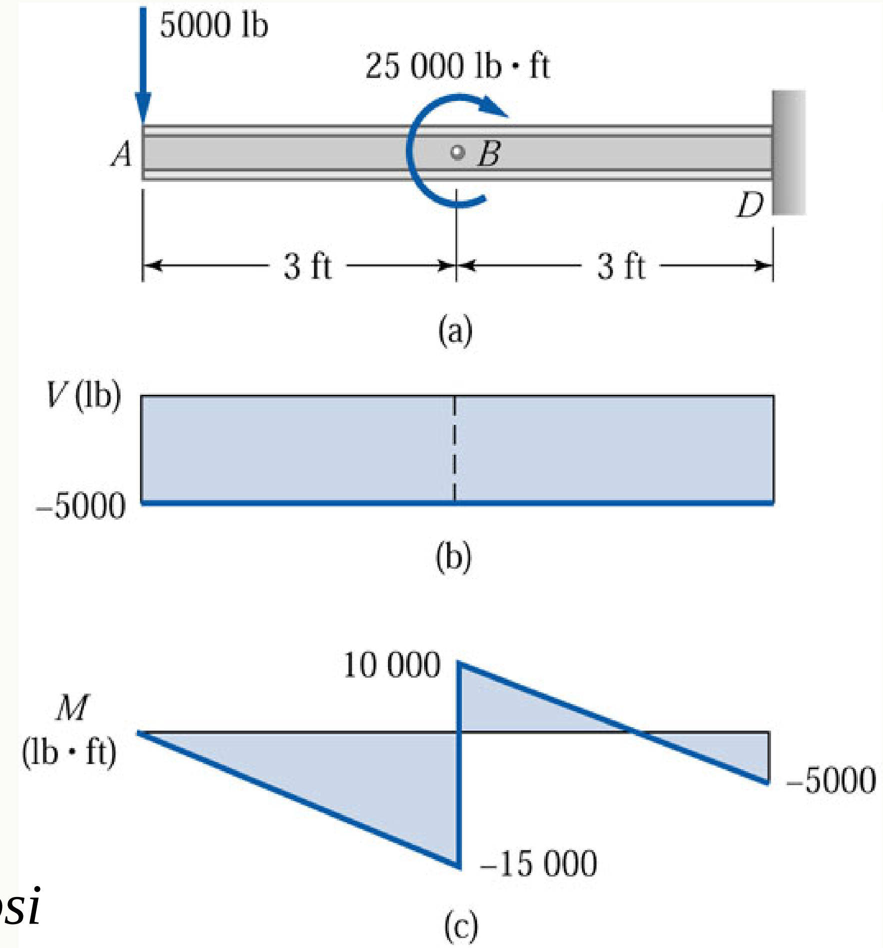
***Sample Problem* 5.4**

The wide- flange section W 14×30 is use as a cantilever beam, as shown in Fig.(a). Find the maximum bending stress in the beam.

Solution

The largest bending moment is $|M_{max}| = 15000\ \text{lb} \cdot \text{ft}$ acting just to the left of section *B*. From the tables in Appendix B, we find that the section modulus of a W14×30 (P520) section is $S = 42.0\ \text{in.}^3$. Therefore, the maximum bending stress in the beam is

$$\sigma_{max} = \frac{|M_{max}|}{S} = \frac{15000 \times 12}{42.0} = 4290\, psi$$



5.3 *Economic Sections*

- The portions of a located near the neutral surface are understressed compared with at the top or bottom. Therefore, beams with certain cross- sectional shape (including a rectangle and circle) utilize the material inefficiently because much of the cross section contributes little to resisting the bending moment.

- Consider, for example, in Fig. 5.5(a) The section modulus has increased to $S = bh^2/6 = 2(6)^2/6 = 12$ in.^{3}. If working stress is $\sigma_w = 18$ ksi, the maximum safe bending moment for the beam is $M = \sigma_w \cdot S = 18\ (12) = 216$ kip·in.

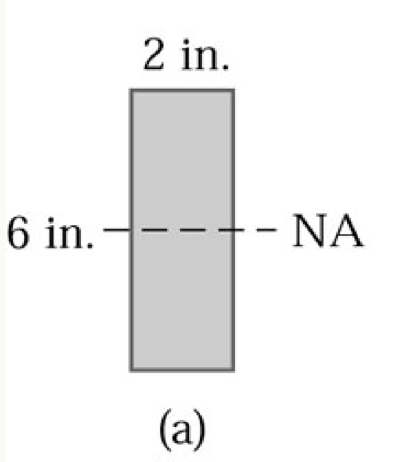
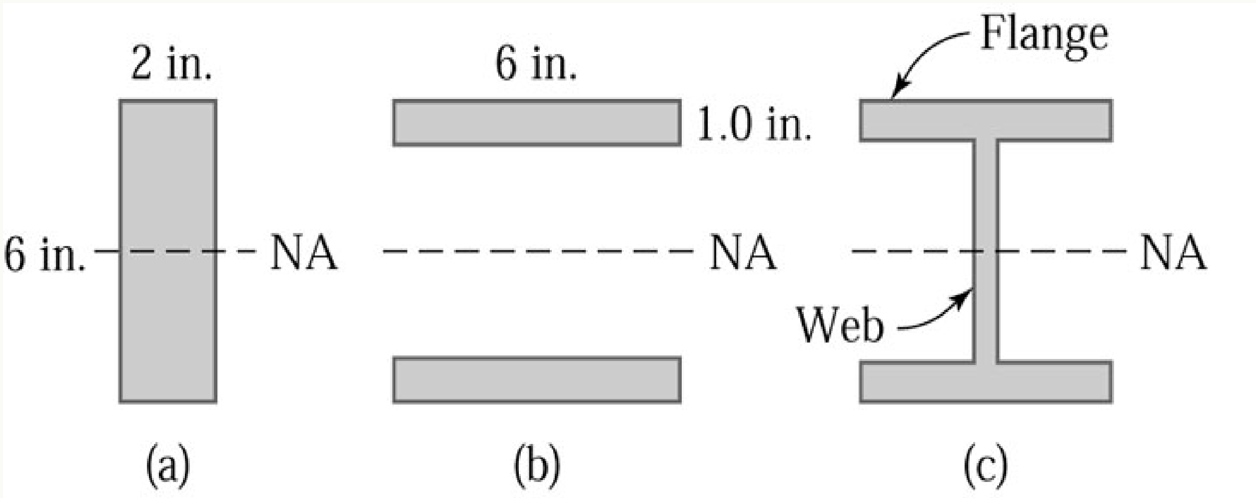


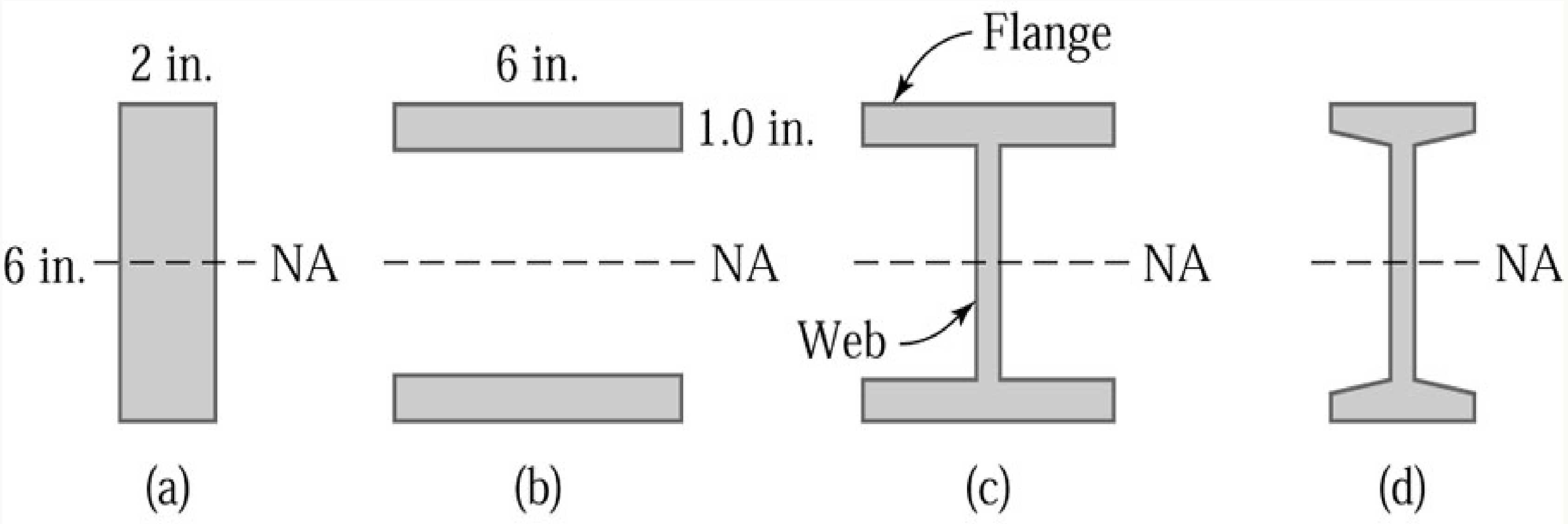
Figure 5.5 Different ways to distribute the 12-in.^{2} cross-sectional area in (a) without changing the depth.

- In Fig. 5.5(b), we have rearranged the area of the cross section but kept the same overall depth. It can be shown that the section that the section modulus has increased to $S = 25.3\ \text{in.}^3$ （the parallel-axis theorem）. Thus, the new maximum allowable moment is $M = 18\,(25.3) = 455\ \text{kip·in.}$, which is more than **twice** the allowable moment for the rectangular section of the same area.
- The section in Fig. 5.5(b) is not practical because its two parts, called the *flanges*. As in Fig. 5.5(c). The vertical connecting piece is known as the *web* of the beam. The web functions as the main shear-carrying component of the beam.



a. *Standard structural shapes*

- Figure 5.5 (c) is similar to a *wide-flange beam*, referred to as a W-shape. Another "slimmer"version of the shape is the I-beam (referred to as an S-shape) shown in Fig. 5.5(d). The I-beam preceded the wide- flange beam, but because it is not as efficient, it has largely been replaced by the wide- flange beam.



- Properties of W-and S-shapes are given in **Appendix B**.

- in SI units, the designation W610×140 indicates a wide-flange beam with a nominal depth of 610mm and a nominal mass per unit length of 140 kg/m. The tables in **Appendix B** indicates the actual depth of the beam is 617 mm and the actual mass is 140.1 kg/m.
- In U.S. Customary units, a W36×300 is a wide-flange beam with a nominal depth 36 in. that weighs 300 lb/ft. The actual depth of this section is 36.74 in.
- Referring to **Appendix B**, in addition to listing the dimensions, tables of structural shapes give properties of the cross-sectional area, such as moment of inertia (I), section modulus (S), and radius of gyration (r)^4 for each principal axis of the area.

When a structural section is selected to be used as a beam. The section modulus must be equal to or greater than section modulus determined by the flexure equation; that is,

$$S \geq \frac{|M_{\max}|}{\sigma_w} \qquad (5.5)$$

the section modulus of the selected beam must be equal to or greater than the ratio of the bending moment to the working stress.

- If a beam is very slender (large L/r), it may fail by *lateral bucking* before the working stress is reached. I-beams are particularly vulnerable to lateral bucking because of their low torsional rigidity and small moment of inertia about the axis parallel to the web.

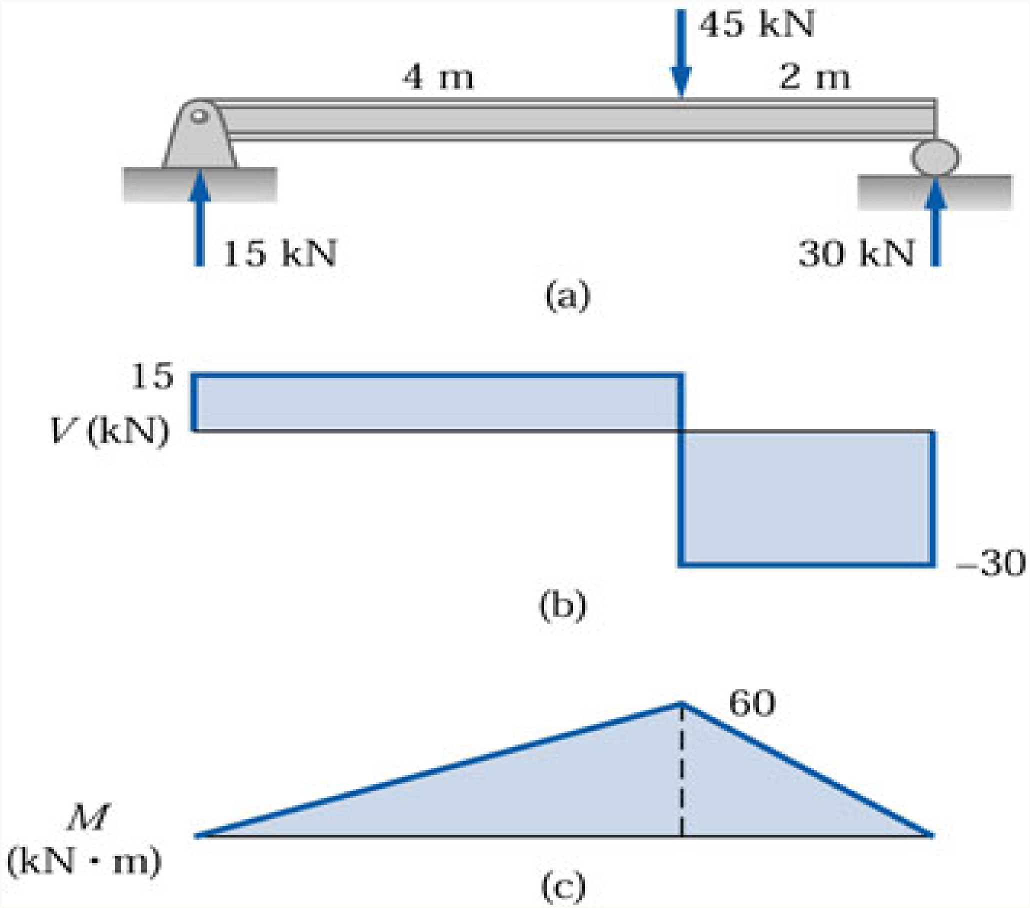
b. *Procedure for selecting standard shapes*

A design engineer is often required to select the **lightest** standard structural shape (such as a W-shape) that can carry a given loading in addition to the weight of the beam. Following is an outline of the selection process;

- Neglecting the weight of the beam, draw the bending moment diagram to find the largest bending moment $M_{max.}$.
- Determine the minimum allowable section modulus from $S_{\text{min}} = |M_{max.}| / \sigma_w$, is the working stress.
- Choose the lightest shape from the list of structural shapes (such as a Appendix B) for which $S \geq S_{\text{min}}$ and note its weight.
- Calculate the maximum bending stress σ_{max} in the selected beam caused by the prescribed loading plus the *weight of the beam*. If $\sigma_{\text{max}} \leq \sigma_w$, the selection is finished. Otherwise, the second-lightest shape with $S \geq S_{\text{min}}$ must be considered and the maximum bending stress recalculated. The process must be repeated unit a satisfactory shape is found.

***Sample Problem* 5.5**

What is the lightest W-shape beam that will support the 45-kN load shown in Fig. (a) without exceeding a bending stress of 120 MPa？Determine the actual bending stress in the beam.



Solution

Finding the reactions shown in Fig.(a), and sketch the shear force and bending moment diagrams in Figs. (b) and (c).

The minimum bending acceptable section modulus that can carry this moment is

$$S_{\min} = \frac{|M_{\max}|}{\sigma_w} = \frac{60 \times 10^3}{120 \times 10^6} = 500 \times 10^{-6} m^3 = 500 \times 10^3 mm^3$$

Referring to the table of Properties of W-shape (Appendix B SI Unit) and find that the following are the lightest beams in each size group that satisfy the requirement $S \geq S_{\min}$: (P508)

Section	S(mm^3)	Mass(kg/m)
W200×52	512×10^3	52.3
W250×45	534×10^3	44.9
W310×39	549×10^3	38.7

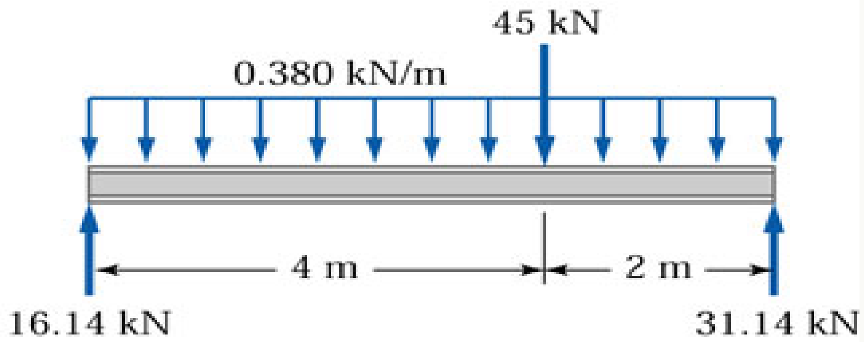
Our first choice is the W310×39 section with S = 549×10^{-6} m^3.

The reason is that although the lightest beam is the cheapest on the basis of the weight alone, headroom clearances frequently require a beam with less depth than the lightest one.

The weight of the beam for the W310×39 section is

$$w_o = (38.7\ \text{kg/m}) \times (9.81\ \text{m/s}^2) = 380\ \text{N/m} = 0.380\ \text{kN/m}$$

From (d) shows the beam supporting **both** the 45-kN load and the weight of the beam. The maximum bending moment is found to be $M_{max} = 61.52\ \text{kN}\cdot\text{m}$, again occurring under the concentrated load.



(d)

Therefore, the maximum bending stress in the selected beam is

$$\sigma_{max} = \frac{|M_{max}|}{S} = \frac{61.52 \times 10^3}{549 \times 10^{-6}} = 112.1 \times 10^6\ pa = 112.1 MPa$$

Because this stress is less than the allowable stress of 120 MPa, the lightest W-shape that can safely support the 45-kN load is

W310×39 (with σ max = 112.1MPa) *Answer*

5.4 *Shear Stress in Beams*

a. *Analysis of flexure action*

- In Fig. 5.6, The separate layers would slide past one another, and the **total bending strength** of the beam would be the sum of the strength of the individual layers. Such **a built-up beam** would be considerably weaker than a **solid beam** of equivalent dimensions.
- From the above observation, we conclude that the horizontal layers in a solid beam are prevented from sliding by shear stresses that act between the layers.

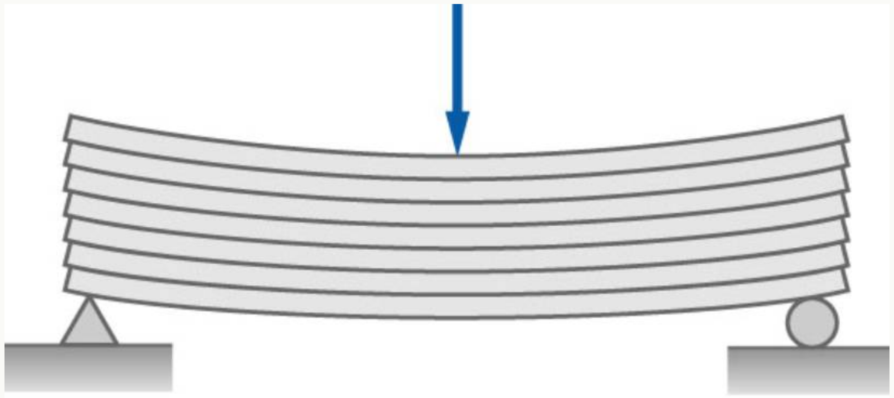


Figure 5.6 Bending of a layered beam with no adhesive between the layers.

- In Fig. 5.7. We isolate the shaded portion of the beam by using two cutting planes: a vertical cut along section 1 and horizontal cut located at the distance *y'* above the neutral axis.

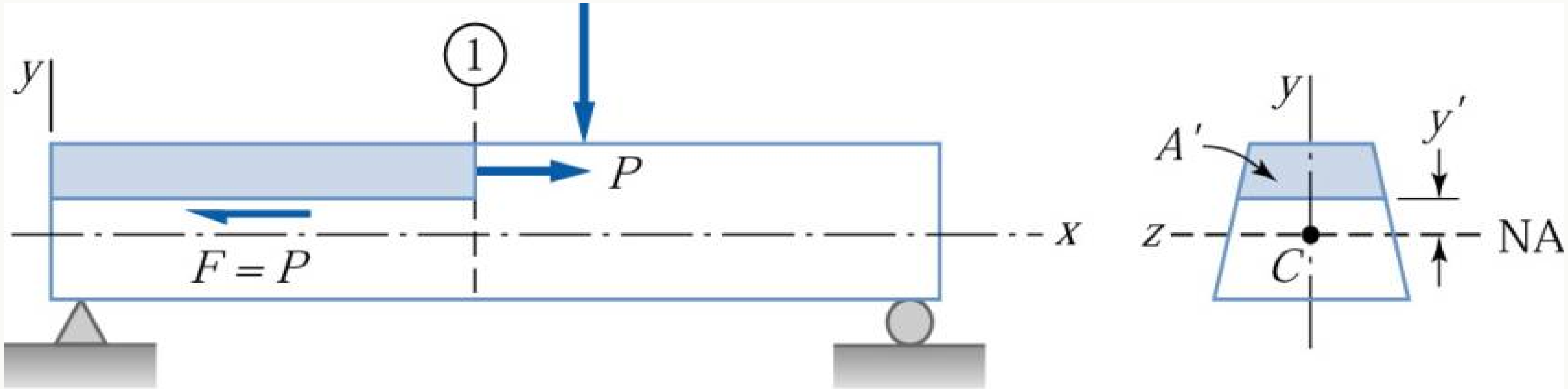


Figure 5.7 Equilibrium of the shaded portion of the beam requires a longitudinal shear force F = P, where P is the resultant of the normal stress acting on area A' of section (1).

- Calculate P using Fig. 5.8. The axial force acting on the area element dA of the cross section is $dP = \sigma\, dA$.
- If M is the bending moment acting at section 1 of the beam, the bending stress is given by Eq. (5.3): $\sigma = -My/I$, where y is the distance of the element from the neutral axis, and I is the moment of inertia of the *entire cross-sectional area* of the beam about the neutral axis.

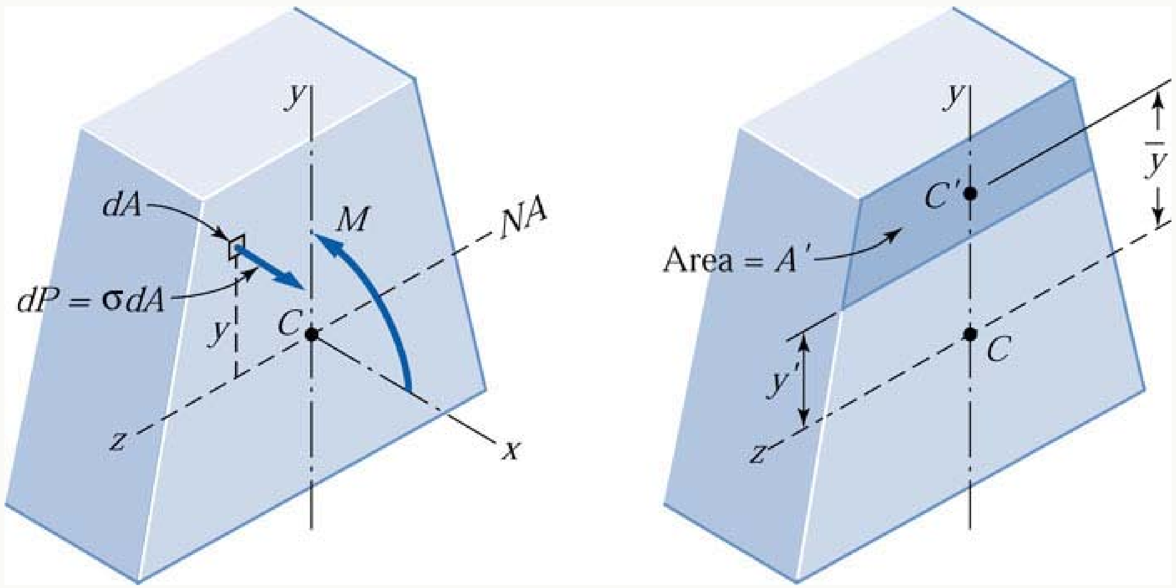


Figure 5.8 Calculating the resultant force of the normal stress over a portion of the cross-sectional area.

$$dP = -\frac{My}{I}dA$$

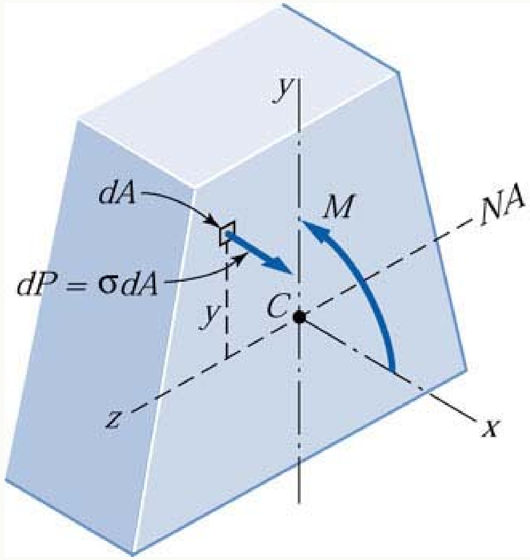
Integrating over the area A', we get

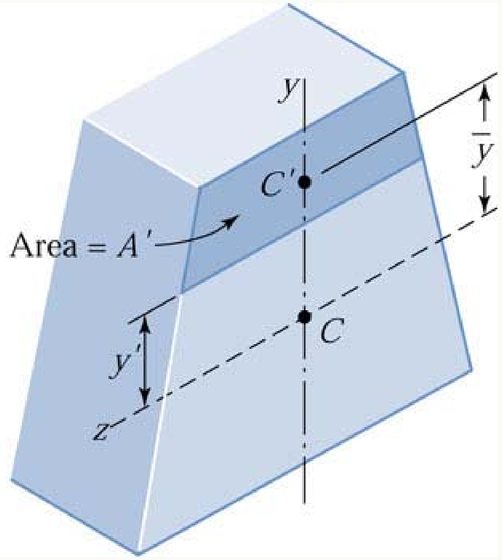
$$P = \int_{A'} dp = -\frac{M}{I}\int_{A'} ydA = -\frac{MQ}{I} \qquad (5.6)$$

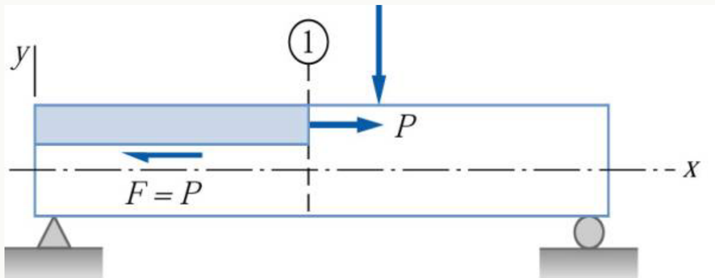
Where

$$\boxed{Q = \int_{A'} ydA} \qquad (5.7a)$$

is the first moment of area *A'* about the neutral axis. The negative sign in Eq. (5.6) indicates that positive *M* results in forces *P* and *F* that are directed opposite to those shown in Fig. 5.7.



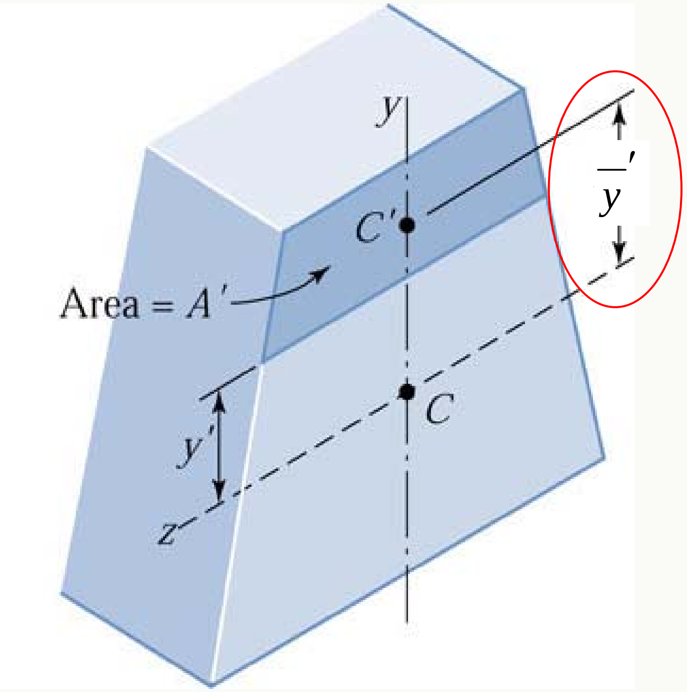




- Denoting the distance between the neutral axis and centroid *C*` of the area *A*’ by $\bar{y}'$, we can write Eq. (5.7) as

$$Q = A`\bar{y}' \qquad (5.7b).$$

- In Eqs. (5.7b), *Q* represents the first moment of the cross-sectional area that lies *above y’*. Because the first moment of the total cross-sectional area about the neutral axis is zero, that first moment of the area *below y’* is - *Q*. Therefore, the magnitude of *Q* can be computed by using the area either **above** or **below** *y’*, whichever is more convenient.



- The maximum value of Q occurs **at the neutral axis** where $y' = 0$. It follows that horizontal shear force F is largest on the neutral surface. The variation of Q with y' for a rectangular cross section is illustrated in Fig. 5.9.

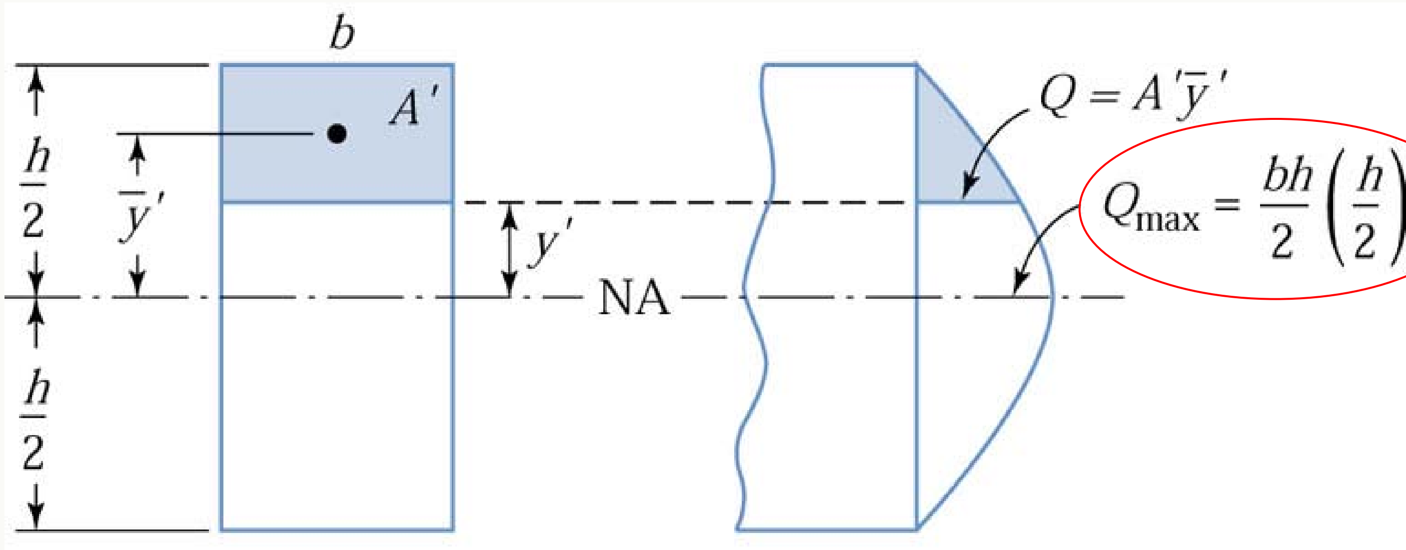


Figure 5.9 Variation of the first moment Q of area A' about the neutral axis for a rectangular cross section.

b. *Horizontal shear stress*

Consider Fig. 5.10. A horizontal plane located a distance *y'* above the neutral axis of the cross section. If the bending moment at section1 of the beam is *M*, the resultant force acting on face 1 of the body is given by Eq. (5.6):

$$\boxed{P = -M\frac{Q}{I}}$$

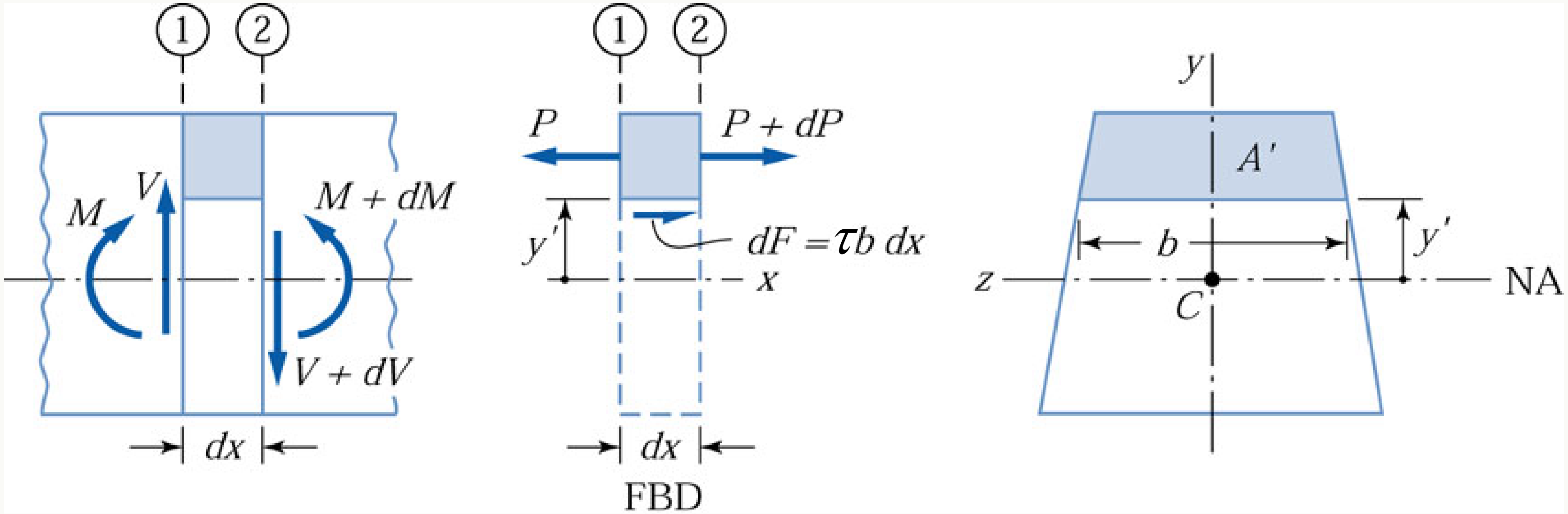


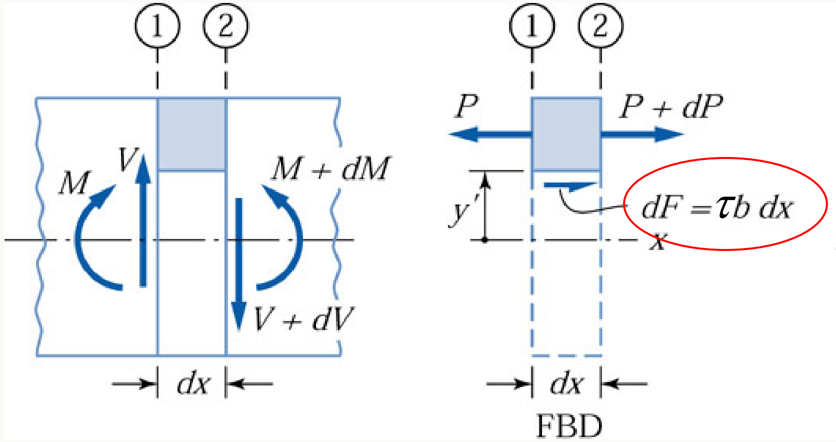
Figure 5.10 Determining the longitudinal shear stress from the free-body diagram of a beam element.

- The bending moment acting at section 2 is $M+dM$, where dM is the infinitesimal change in M over the distance dx. Therefore, the resultant normal force acting on face 2 of the body is

$$p+dP=-(M+dM)\frac{Q}{I} \qquad P=-M\frac{Q}{I}$$

$$(P+dP)-P=-(M+dM)\frac{Q}{I}-\left(-M\frac{Q}{I}\right)$$

$$=-dM\frac{Q}{I}$$



- Equilibrium can exist only if there is an equal and opposite shear force dF acting on the horizontal surface. If we let τ be the *average shear stress* acting on the horizontal surface, its resultant is $dF = \tau\, bdx$. Where b is the width of the cross section at $y = y$`, as shown in Fig. 5.10. The equilibrium requirement for the horizontal forces is

$$\sum F=0: \ (P+dP)-P+\tau\, b\, dx=0$$

- Substituting for$(P+dP)$ - P from Eq. (a), we get

$$-dM\frac{Q}{I}+\tau b dx=0 \qquad \tau=\frac{dM}{dx}\frac{Q}{Ib} \tag{b}$$

Recalling the relationship $V = dM/dx$ between the shear force and the bending moment we obtain for the average horizontal shear stress τ

$$\boxed{\tau=\frac{VQ}{Ib}} \tag{5.8}$$

c. Vertical shear stress

- Eq. (5.8) $\tau = \dfrac{VQ}{Ib}$ (a plane parallel to the neutral surface). A shear stress is always accompanied by a complementary shear stress of equal magnitude, the two stresses acting on mutually perpendicular plane.

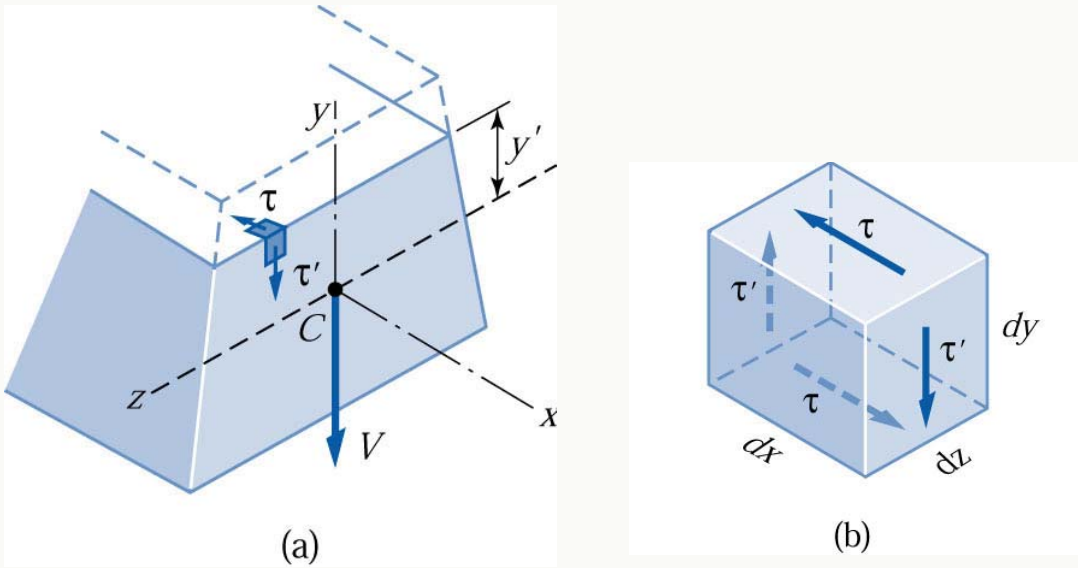
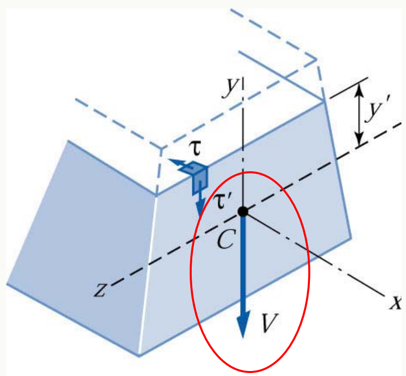


Figure 5.11 The vertical stress τ' acting at a point on a cross section equals the longitudinal shear stress τ acting at the same point.

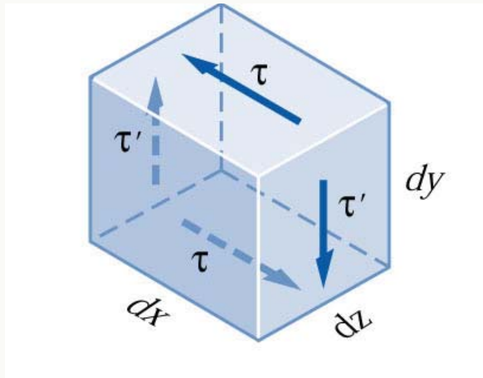
- In a beam, the complementary stress τ ' is a vertical shear stress that acts on the cross section of the beam, as illustrated in Fig. 5.11 (a). Because $\tau = \tau$ ', Eq.(5.8) can be used to compute the vertical as well as the horizontal shear stress at a point in a beam.

- The resultant of the vertical shear stress on the cross-sectional area *A* of the beam is,of course, the shear force *V*;. $V = \int_A \tau dA$



(a)

- To prove that $\tau = \tau$ `, consider Fig. 5.11(b). The horizontal and vertical forces are $\tau\, dxdz$ and $\tau$ '$dydz$, respectively. These forces from two couples of opposite sense. For rotational equilibrium, the magnitudes of the couples must be equal; that is, ($\tau\, dxdz$) $dy$ = ($\tau$ `$dydz$) dx, which yields $\tau = \tau$ '.



(b)

d. Discussion and limitations of the shear stress formula

- The shear stress formula $\tau = VQ/(Ib)$ predicts that the largest shear stress in a prismatic beam occurs at the cross section that carries the largest vertical shear force V.
- The location (the value of y') of the maximum shear stress within that section is determined by the ratio Q/b. Because Q is always maximum at $y' = 0$, the neutral axis is usually a candidate for the location of the maximum shear stress.
- However, If the width b at the neutral axis is larger than at other parts of the cross section, it is necessary to compute τ at two or more values of y' before its maximum value can be determined.

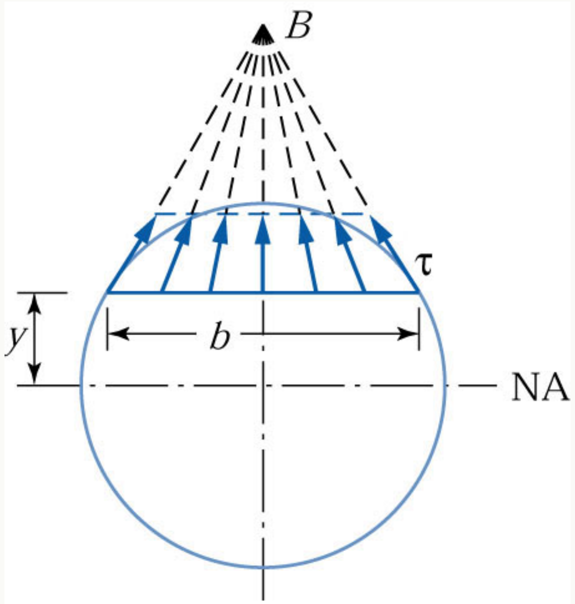
- When deriving the shear stress formula, Eq. (5.8), $\tau = \dfrac{VQ}{Ib}$ τ should be considered at the ***average*** shear stress. This restriction is necessary because the variation of the shear stress across the width *b* the cross section is often unknown.

- Equation (5.8) is sufficiently accurate for rectangular cross sections and for cross sections that are composed of rectangles, such as W and S-shapes.

- Let us consider as an example the **circular cross** section in Fig. 5.12.

$$\boxed{\tau = \frac{VQ}{Ib}}$$

Figure 5.12 Shear stress distribution along a horizontal line of a circular cross section.



- For other cross- sectional shapes, however, the formula for τ must be applied with caution. Let us consider as an example the circular cross section in Fig. 5.12.

- It can be shown that the shear stress at the periphery of the section must be tangent to the boundary, as shown in the figure.

- The direction of shear stresses at interior points is unknown, except at the centerline, where the stress is vertical due to symmetry. To obtain an estimate of the maximum shear stress, the stresses are assumed to be directed toward a common center *B*, as shown.

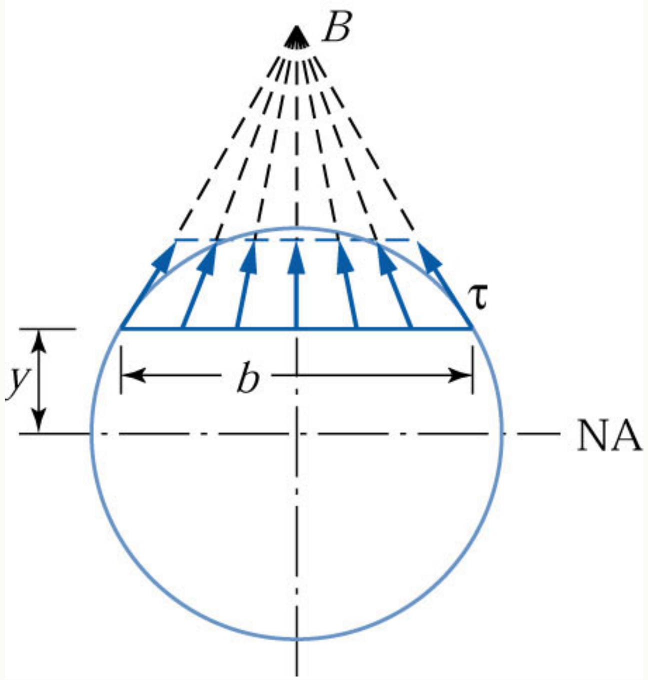


Figure 5.12 Shear stress distribution along a horizontal line of a circular cross section.

- The vertical components of these shear stresses are assumed to be **uniform** across the width of the section and are computed from Eq. (5.8). Under this assumption, **the shear stress at the neutral axis is 1.333$V/(\pi r^2)$. (4/3)($V/\pi r^2$)**

- A more elaborate analysis shows that the shear stress actually varies from 1.23 $V/(\pi r^2)$ at the edges to 1.38 $V/(\pi r^2)$ at the center.

- Shear stress, like normal stress, exhibits stress concentrations near **shape corners**, **fillets** and **holes** in the cross section. The junction between the web and the flange of a W-shape is also an area of stress concentration.

e. *Rectangular and wide-flange sections*

Determine the shear stress as a function of *y* for a rectangular cross section of base *b* and height *h*. From Fig. 5.13, the shaded area is $A' = b\,[(h/2)-y]$, its centroidal coordinate being $\bar{y}' = [(h/2)+y]/2$.

$$Q = A'\bar{y}' = \left[b\left(\frac{h}{2}-y\right)\right]\left[\frac{1}{2}\left(\frac{h}{2}+y\right)\right] = \frac{b}{2}\left(\frac{h^2}{4}-y^2\right)$$

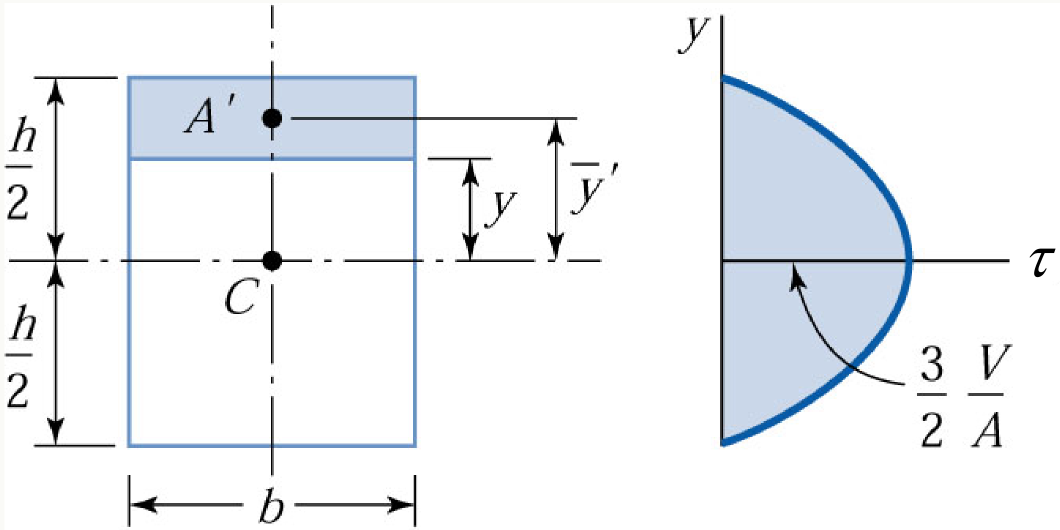


Figure 5.13 Shear stress distribution on a rectangular cross section.

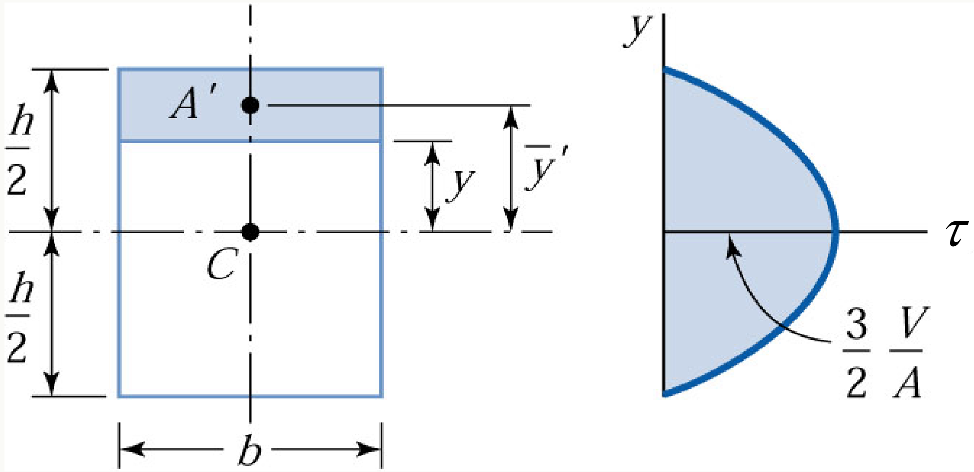
$$\boxed{\tau = \frac{VQ}{Ib} = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)} \qquad \text{(c)}$$

The shear stress is distributed **parabolically** across the depth of the section, as shown in Fig.5.13. The maximum shear stress occurs at the neutral axis. If we substitute $y = 0$ and $I = bh^3/12$, Eq. (c) reduces to

$$\boxed{\tau_{\max} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A}} \qquad (5.9)$$

where A is the cross –sectional area.

The shear stress in rectangular section is 50% greater than the average shear stress on the cross section.



- In wide-flange sections (W-shapes), most of the bending moment is carried by the flanges, whereas the web resists the bulk of the vertical shear force. Figure 5.14. Q is contributed mainly by the flanges of the beam. Consequently, Q does not vary with y, so that the shear stress in the web is almost constant.

- In fact $\tau_{max} = V/A_{web}$ can be used as an approximation to the maximum shear stress in most cases, where A_{web} is the cross-sectional area of the web.

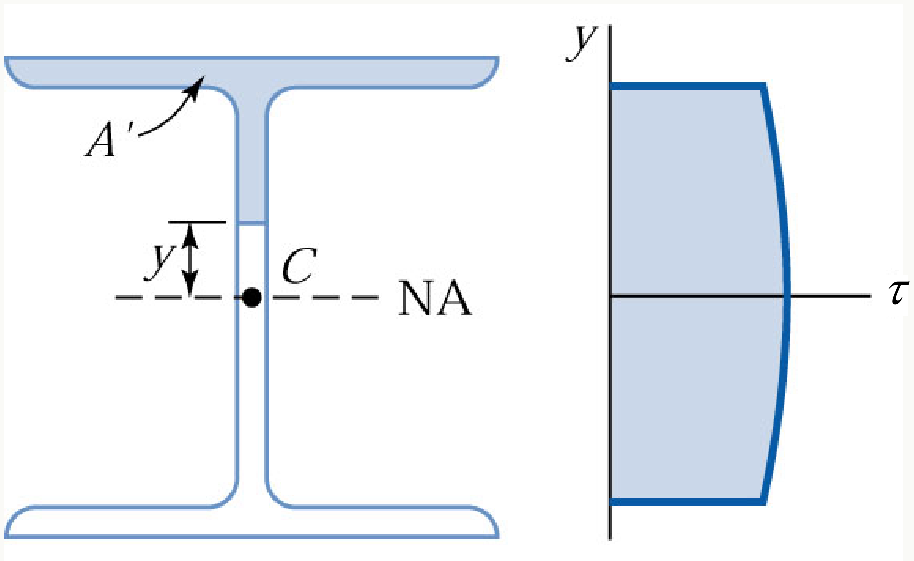


Figure 5.14 Shear Stress distribution on the web of a wide-flange beam.

a. *Procedure for analysis of shear stress*:

- Use equilibrium analysis to determine the vertical shear force *V* acting on the cross section containing the specified point (the construction of a shear force diagram is usually a good idea).

- Locate the neutral axis and compute the moment of inertia *I* of the cross- sectional area about the neutral axis (If the beam is a standard structural shape, its cross- sectional properties are listed in Appendix B.)

- Compute the first moment Q of the cross- sectional area that lies above (or below)the specified point.

- Calculate the shear stress from $\tau = VQ/(Ib)$, where *b* is the width of the cross section at the specified point.

- The maximum shear stress τ_{max} on a given cross section occurs where Q/b is largest.

- If the width b is constant, then τ_{max} occurs at the neutral axis because that is where Q has its maxmum value.

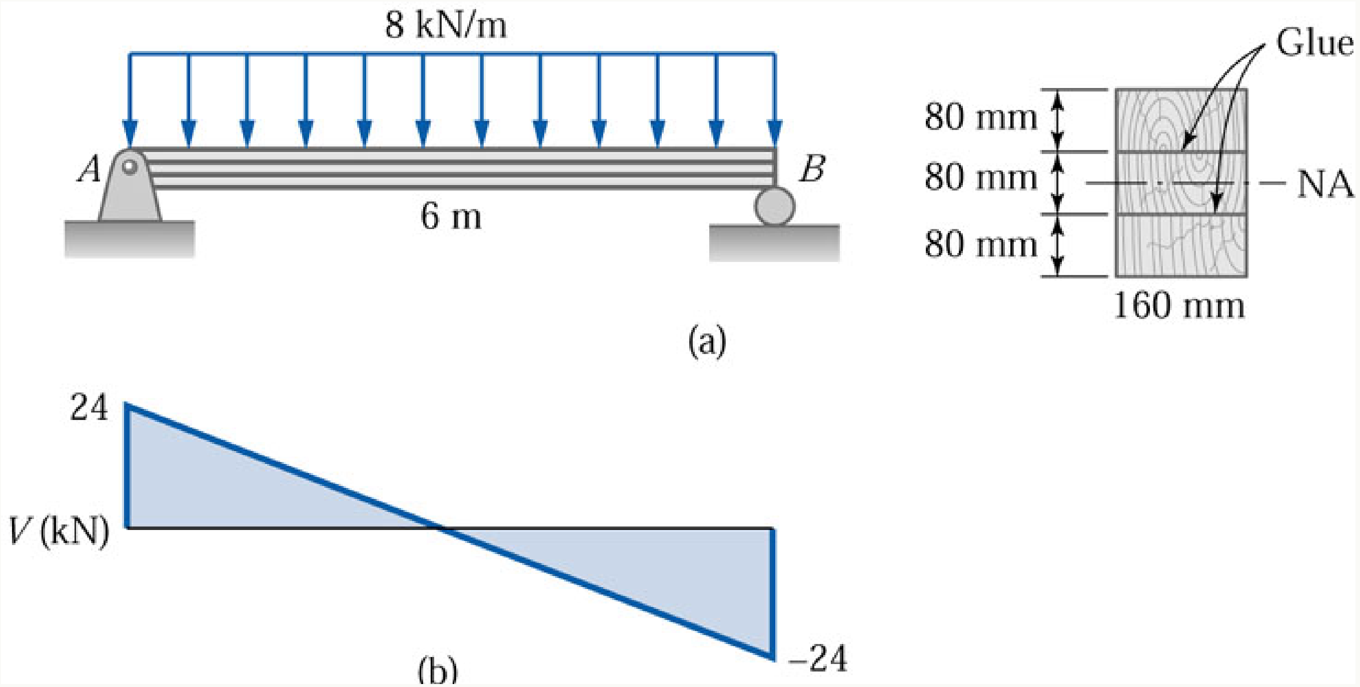
- If b is not constant, it is necessary to compute the shear stress at more than one point in order to determine its maximum value.

In the U.S. Customary system, $\tau\left[lb/in^2.\right] = \dfrac{V[lb]Q\left[in.^3\right]}{I\left[in.^4\right]b[in.]}$

In the SI system, $\tau\left[N/m^2\right] = \dfrac{V[N]Q\left[m^3\right]}{I\left[m^4\right]b[m]}$

***Sample Problem* 5.6**

The simply supported wood beam in Fig.(a) is fabricated by gluing together **three** 160-mm by 80-mm plans as shown. Calculate the maximum shear stress in (1) the glue; and (2) the wood.



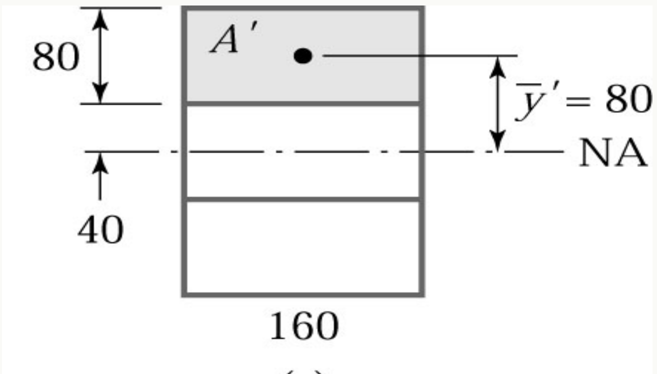
Solution

From the shear force diagram in Fig. (b), the maximum shear force in the beam is $V_{max} = 24$ kN, occurring at the supports. The moment of inertia of the cross-sectional area of the beam about the neutral axis is

$$I = \frac{bh^3}{12} = \frac{160(240)^3}{12} = 184.32 \times 10^6 \, mm^4 = 184.32 \times 10^{-6} \, m^4$$

Part 1

The shear stress is the glue corresponds to the horizontal shear stress. Its maximum value can be computed from Eq. (5.8): $\tau_{max} = V_{max} Q/(Ib)$, where Q is the first moment of the area A' shown in Fig.(c); that is,



(c)

$$Q = A'\bar{y}' = (160 \times 80)(80) = 1.024 \times 10^6 = 1.024 \times 10^{-3} \, m^3$$

Therefore, the shear stress in the glue, which occurs over either support, is

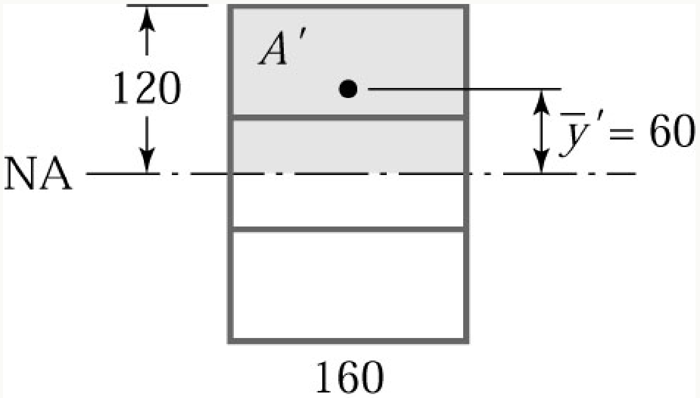
$$\tau_{\max} = \frac{V_{\max} Q}{Ib} = \frac{\left(24\times10^{3}\right)\left(1.024\times10^{-3}\right)}{\left(184.32\times10^{-6}\right)(0.160)}$$

$$= 8.33\times10^{3}\,Pa = 8.33kPa$$

Answer

Part 2

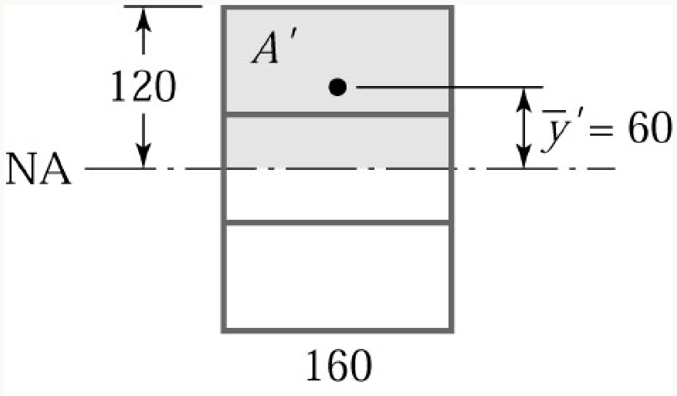
Because the cross section is rectangular, the maximum shear stress in the wood can be calculated from Eq. (5.9):



(d)

$$\tau_{\max} = \frac{3}{2}\frac{V_{\max}}{A} = \frac{3}{2}\frac{\left(24\times10^{3}\right)}{(0.160)(0.240)} = 938\times10^{3}\,Pa = 938kPa$$

The same result can be obtained from Eq. (5.8), where now *A’* is the area above the neutral axis, as indicated in Fig. (d). The first moment of this area about the neutral axis is



(d)

$$Q = A'\bar{y}' = (160 \times 120)(60) = 1.152 \times 10^{6}\,mm^{3} = 1.152 \times 10^{-3}\,m^{3}$$

Equation (5.8)this becomes

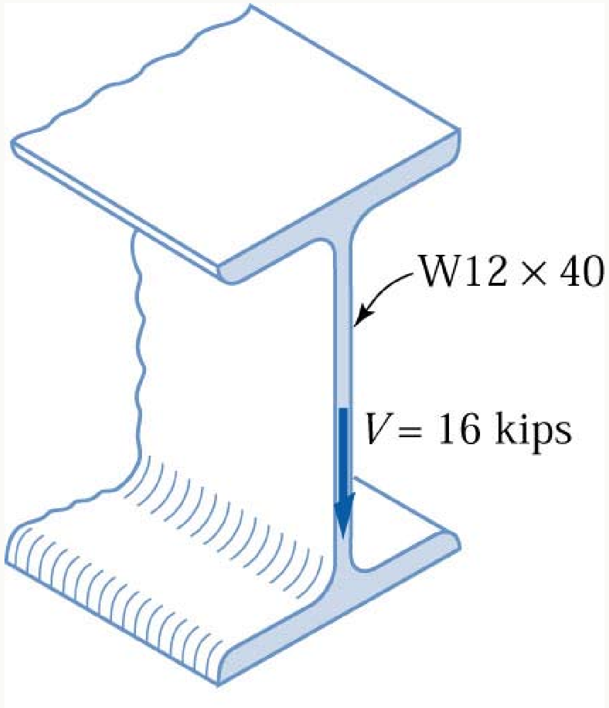
$$\tau_{\max} = \frac{V_{\max} Q}{Ib} = \frac{(24 \times 10^{3})(1.152 \times 10^{-3})}{(184.32 \times 10^{-6})(0.160)}$$

$$= 938 \times 10^{3}\,Pa = 938\,kPa$$

which agrees with the previous result.

***Sample Problem* 5.7**

The W12×40 section in Fig.(a) is used as a beam. If the vertical shear acting at a certain section of the beam is 16 kips, determine the following at that section: (1) the minimum shear stress in the web;(2) the maximum shear stress in the web; and (3)the percentage of the shear force that is carried by the web.



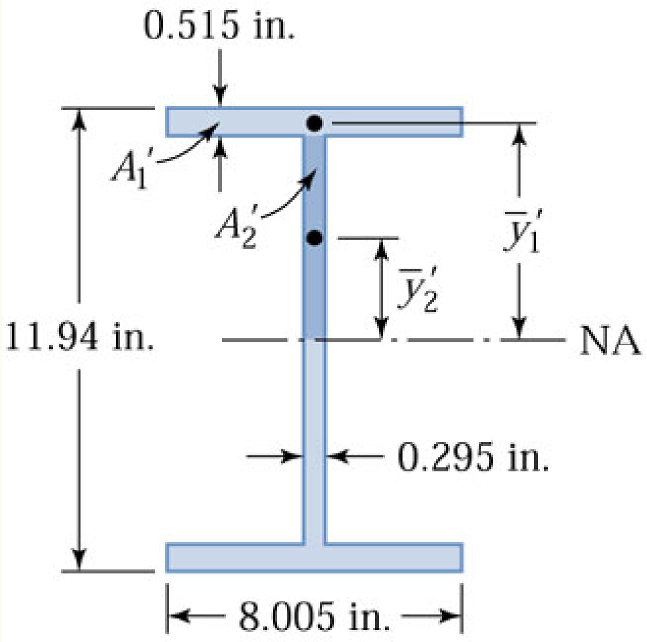
(a)

Solution

The W12×40 section is shown in Fig.(b), where the dimensions were obtained from the tables in **Appendix B** (P521). The drawing approximates the web and the flanges by rectangles, thereby ignoring the small fillets and rounded corners present in the actual section. The **tables** also list the moment of inertia of the section about the neutral axis as $I = 310 \text{ in.}^4$.

Part 1

The minimum shear stress in the web occurs at the junction with **the flange**, where Q/b is **smallest** (note that $b = 0.295$ in. is constant within the web). Q is the first moment of the area $A'1$ shown in Fig.(b) about the neutral axis:



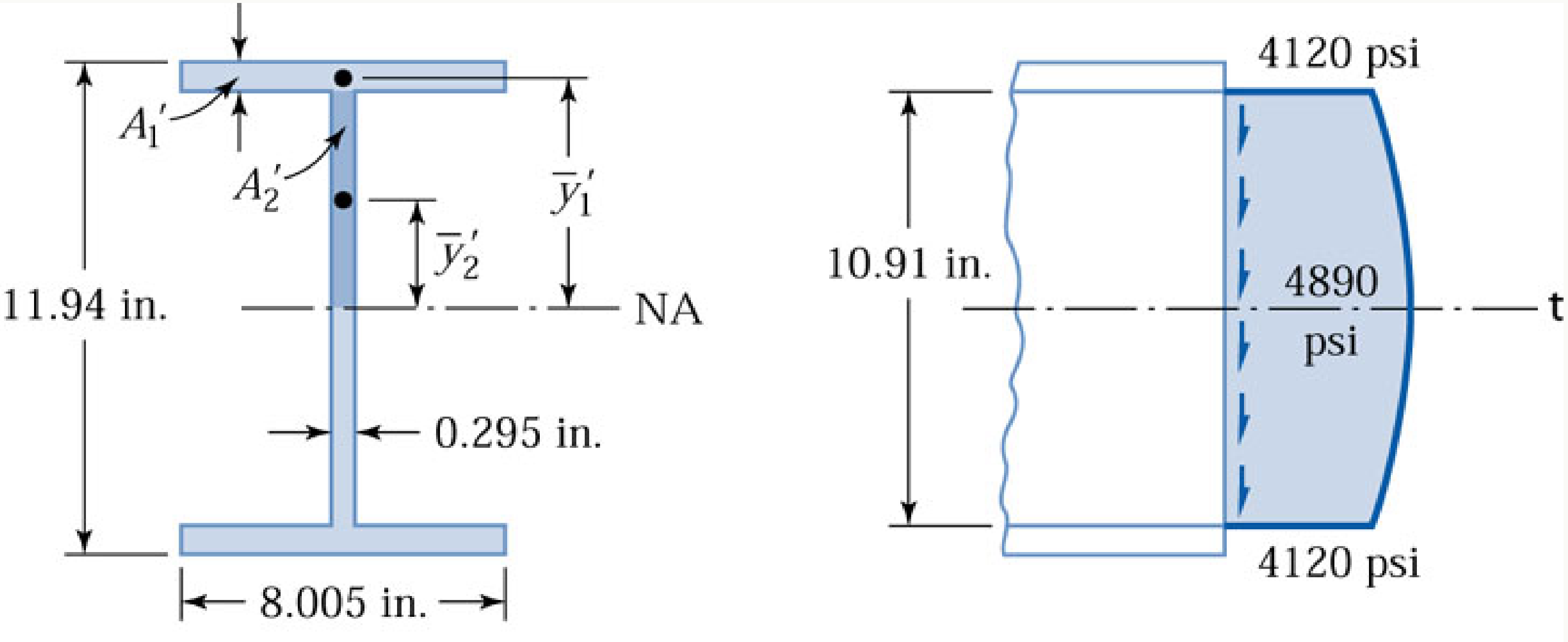
(b)

$$Q = A'_1 \bar{y}' = (8.005 \times 0.515)\frac{11.94 - 0.515}{2} = 23.55 in.^3$$

The minimum shear stress in thus becomes

$$\tau_{\min} = \frac{VQ}{Ib} = \frac{(16\times10^3)(23.55)}{(310)(0.295)} = 4120\ psi$$

Answer

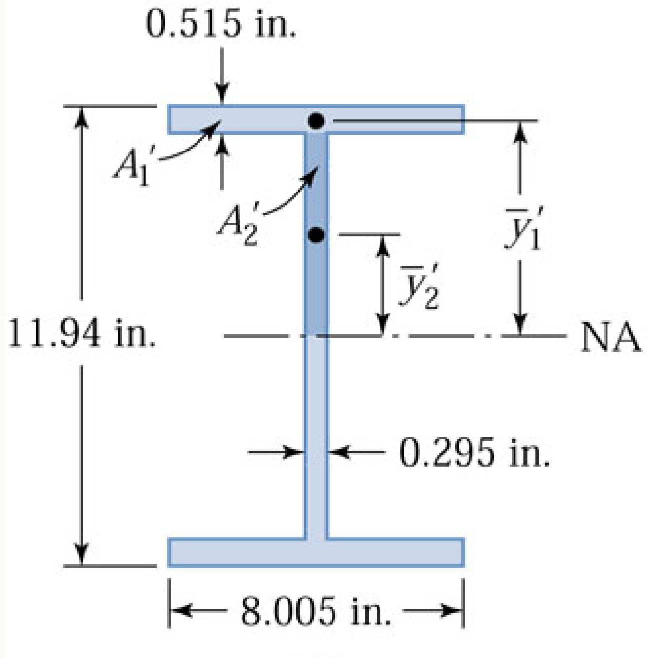


(b) (c)

Part 2

The maximum shear stress is located **at the neutral axis**, where Q/b is **largest**. Hence, Q is the first moment of the area above (or below) the neutral axis.

The moment of A'_1 was calculated in part 1. The moment of A'_2 about the neutral axis is where



(b)

$$A`_1\bar{y}` = (8.005\times 0.515)\frac{11.94-0.515}{2} = 23.55 in.^3$$

$$= 23.55 in.^3$$

$$A`_2 = \left(\frac{11.94}{2}-0.515\right)(0.295) = 1.6092 in.^2$$

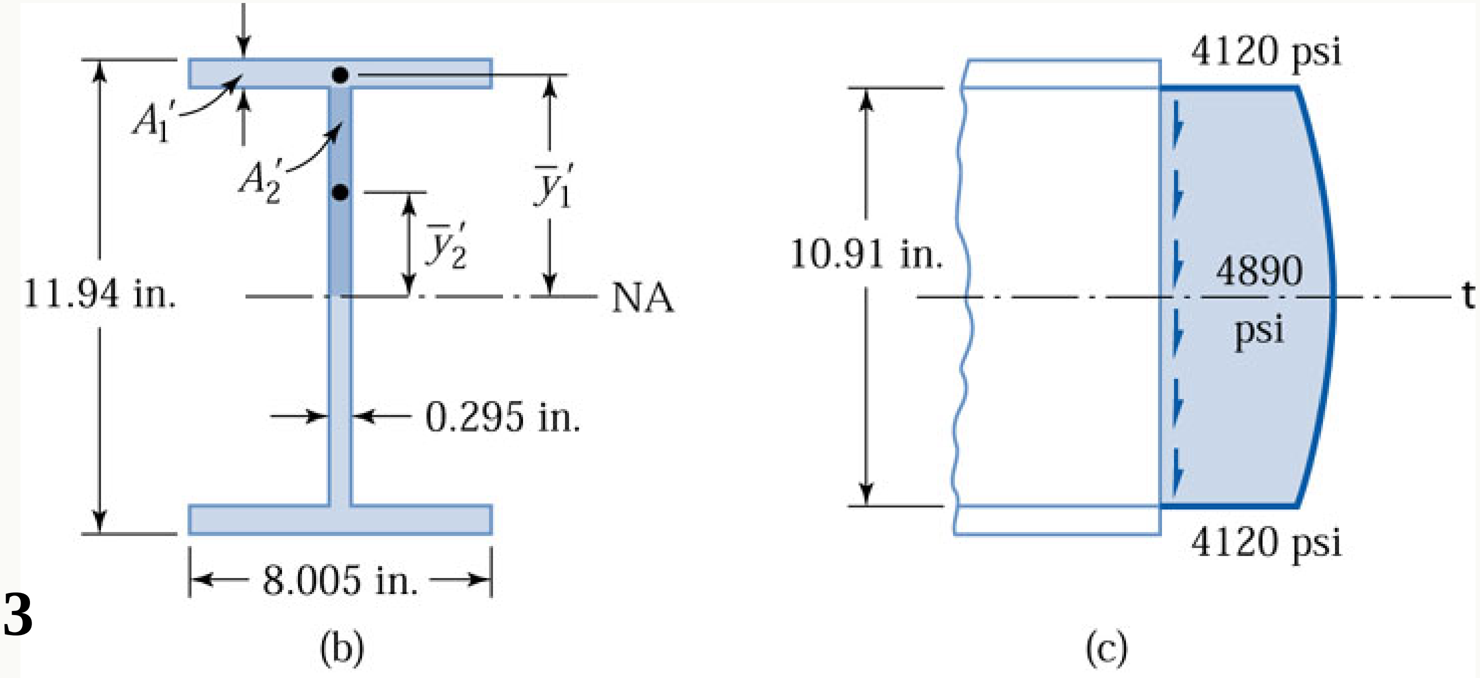
$$\bar{y}`_2 = \frac{1}{2}\left(\frac{11.94}{2}-0.515\right) = 2.7275 in.$$

$$Q = A`_1\bar{y}`_1 + A`_2\bar{y}`_2 = 23.55 + (1.6092)(2.7275) = 27.94 in.^3$$

The maximum shear stress in the web becomes

$$\tau_{\max} = \frac{VQ}{Ib} = \frac{(16\times 10^3)(27.94)}{(310)(0.295)} = 4890 \ psi$$

Answer



Part 3

The distribution of the shear stress in the web is shown in Fig.(c). The shear force carried by the web is

$$V_{\text{web}} = (\text{cross section area of web}) \times (\text{area of shear diagram})$$

The shear stress distribution is parabolic. Recalling that the area of a parabola is (2/3) (base × height).

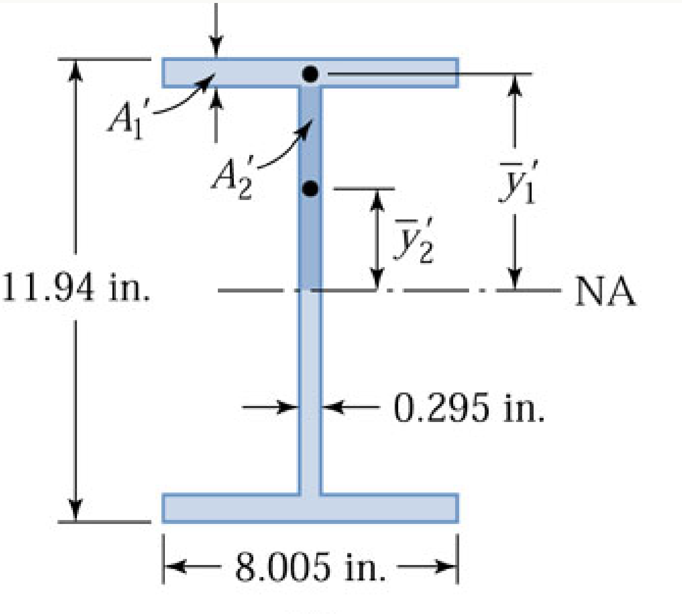
$$V_{web} = (10.91 \times 0.295)\left[4120 + \frac{2}{3}(4890 - 4120)\right] = 14910 lb$$

Therefore the percentage of the shear force carried by the web is

$$\frac{V_{web}}{V}\times 100\% = \frac{14910}{16000}\times 100\% = 93.2\% \qquad \textbf{\textit{Answer}}.$$

The result confirms that the flanges are ineffective in resisting the vertical shear

It was mentioned in Art. 5.5 that we can use $\tau_{max} = V/A_{web}$ as a rough approximation for the maximum shear stress.



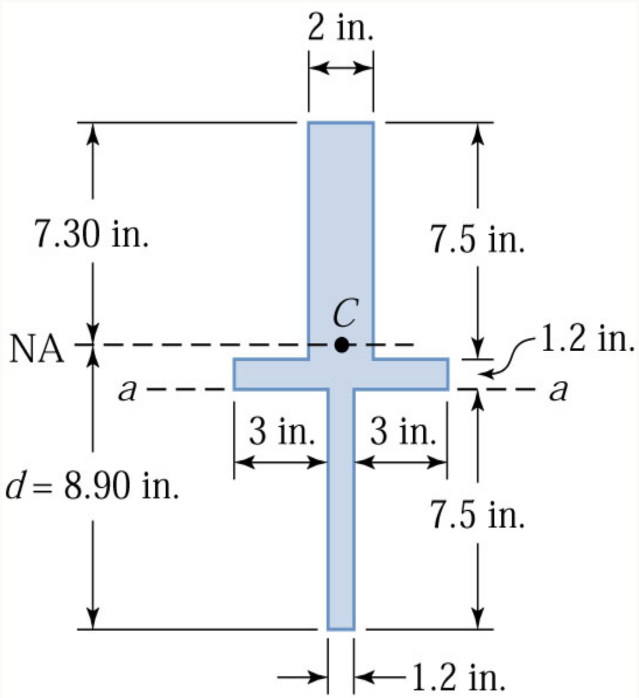
(b)

$$\frac{V}{A_{web}} = \frac{16\times 10^3}{(10.91)(0.295)} = 4970\ psi$$

which differs from $\tau_{max} = 4890$ psi computed in Part 2 by less than 2%.

***Sample Problem* 5.8**

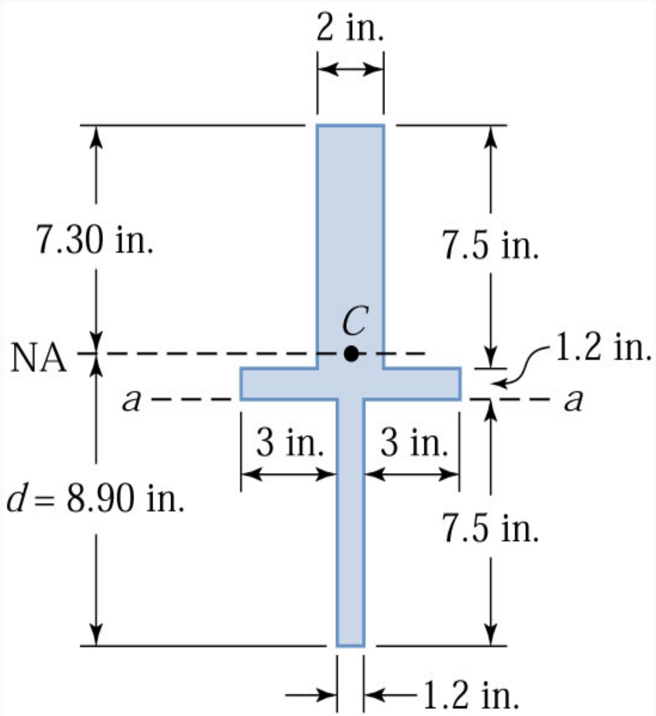
The figure shows the cross section of a beam that carries a vertical shear force V = 12 kips. The distance from the bottom of the section to the neutral axis is d = 8.90 in., and the moment of inertia of the cross –sectional area about the neutral axis is I = 547 in.^4. Determine the maximum shear stress on this cross section.



Solution

The maximum shear stress may occur of the neutral axis (where Q is largest) or at level *a-a* in the lower fin (where the width of the cross section is smaller than at the neutral axis).

Shear Stress at Neutral Axis Take Q to be the first moment of the rectangular area ***above*** the neutral axis (the area below the neutral axis could also be used).



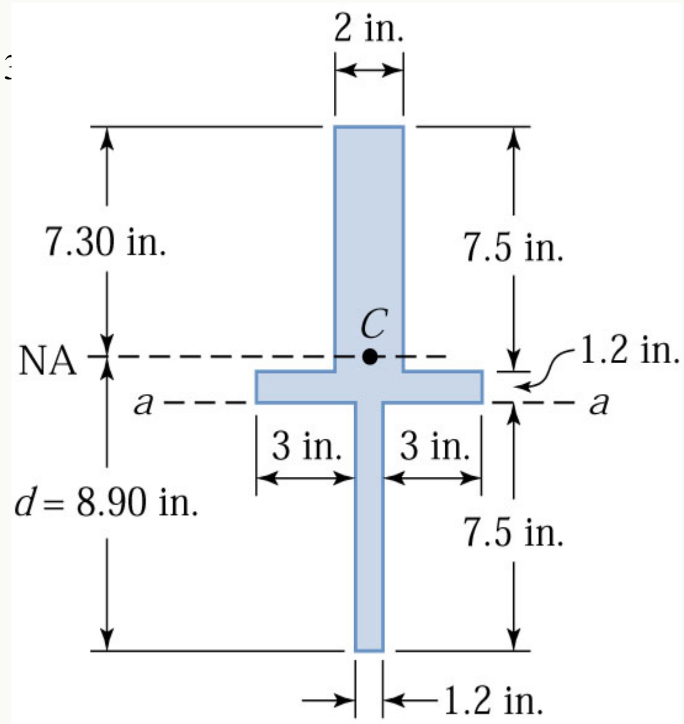
$$Q = A\bar{y} = (2 \times 7.30)\frac{7.30}{2} = 53.29 in.^3$$

and the shear stress at the neutral axis is

$$\tau = \frac{VQ}{Ib} = \frac{(12 \times 10^3)(53.29)}{(547)(2)} = 58.5\,psi$$

Shear Stress at a-a It is easier to compute Q by using the area *below* the line *a-a* rather than the area above the line. The dimensions of this area are b =1.2 in. and h =7.5 in. Consequently,

$$Q = A\bar{y} = (1.2\times 7.5)\left(8.90 - \frac{7.5}{2}\right) = 46.35 in.^3$$



and the shear stress becomes

$$\boxed{\tau = \frac{VQ}{Ib} = \frac{(12\times 10^3)(46.35)}{(547)(1.2)} = 847\, psi}$$

Shear Stress at Neutral Axis

$$\boxed{\tau = \frac{VQ}{Ib} = \frac{(12\times 10^3)(53.29)}{(547)(2)} = 58.5\, psi}$$

The maximum shear stress is the largest of the two value;

$\tau_{\max} =$ 847 psi (occurring at *a-a*) ***Answer***